Hilbert functions of points on Schubert varieties in Orthogonal Grassmannians

K. N. Raghavan
Institute of Mathematical Sciences
C. I. T. Campus, Taramani
Chennai 600 113, INDIA
email: knr@imsc.res.in

Shyamashree Upadhyay
Chennai Mathematical Institute
Plot No. H1, SIPCOT IT Park
Padur Post, Siruseri 603 103, Tamilnadu, INDIA
email: shyama@cmi.ac.in

04 April 2007

Abstract

A solution is given to the following problem: how to compute the multiplicity, or more generally the Hilbert function, at a point on a Schubert variety in an orthogonal Grassmannian. Standard monomial theory is applied to translate the problem from geometry to combinatorics. The solution of the resulting combinatorial problem forms the bulk of the paper. This approach has been followed earlier to solve the same problem for the Grassmannian and the symplectic Grassmannian.

As an application, we present an interpretation of the multiplicity as the number of non-intersecting lattice paths of a certain kind.

Taking the Schubert variety to be of a special kind and the point to be the "identity coset," our problem specializes to a problem about Pfaffian ideals treatments of which by different methods exist in the literature. Also available in the literature is a geometric solution when the point is a "generic singularity."

Mathematics Subject Classification 2000: 05E15 (Primary), 13F50, 13P10, 14L35 (Secondary)

Contents

Introduction

In this paper the following problem is solved: given a Schubert variety in an *orthogonal Grassmannian* (by which is meant the variety of isotropic subspaces of maximum possible dimension of a finite dimensional vector space with a symmetric non-degenerate form—see §1 for precise definitions) and an arbitrary point on the Schubert variety, how to compute the multiplicity, or more generally the Hilbert function, of the local ring of germs of functions at that point. In a sense, our solution is but a translation of the problem: we do not give closed form formulas but *alternative* combinatorial descriptions. The meaning of "alternative" will presently become clear.

The same problem for the Grassmannian was treated in [11, 8, 7, 9, 12] and for the symplectic Grassmannian in [4]. The present paper is a sequel to [11, 7, 9, 12, 4] and toes the same line as them. In particular, its strategy is borrowed from them and runs as follows: first translate the problem from geometry to combinatorics, or, more precisely, apply *standard monomial theory* to obtain an *initial* combinatorial description of the Hilbert function (the earliest version of the theory capable of handling Schubert varieties in an orthogonal Grassmannian is to be found in [17]); then transform the initial combinatorial description to obtain the desired alternative description. But that is easier said than done.

While the problem makes sense for Schubert varieties of any kind and standard monomial theory itself is available in great generality [13, 15], the translation of the problem from geometry to combinatorics has been made—in [14]—only for "minuscule[1] generalized Grassmannians." Orthogonal Grassmannians being minuscule, this translation is available to us and we have an initial combinatorial description of the Hilbert function. As to the passage from the initial to the alternative description—and this is where the content of the present paper lies—neither the end nor the means is clear at the outset.

The first problem then is to *find* a good alternative description. But how to measure the worth of an alternative description? The interpretation of multiplicity as the number of certain non-intersecting lattice paths (deduced in §11 from our alternative description) seems to testify to the correctness of our alternative description, but we are not sure if there are others that are equally or more correct.

The proof of the equivalence of the initial and alternative combinatorial descriptions is, unfortunately, a little technically involved. It builds on the details of the proofs of the corresponding equivalences in the cases of the Grassmannian and the symplectic Grassmannian. In [10] it is shown that the equivalence in the case of the Grassmannian is a kind of KRS correspondence, called "bounded KRS." The proof there is short and elegant and it would be nice to realise the main result of the present paper too in a similar spirit as a kind of KRS correspondence.

The initial description is in terms of "standard monomials" and the alterna-

[1]Symplectic Grassmannians are not minuscule but can be treated as if they were.

tive description in terms of "monomials in roots." The equivalence of the two descriptions thus gives a bijective correspondence between standard monomials and monomials in roots. Roughly—but not actually—the correspondence maps each standard monomial to its initial term (with respect to a certain monomial order). Thus it is natural to wonder whether we can compute the initial ideal of the ideal of the tangent cone to the Schubert variety at the given point. We believe that this can be done but that it is far more involved and difficult than the corresponding computation for Grassmannians and symplectic Grassmannians (the natural set of generators of the ideal of the tangent cone do not form a Gröbner basis unlike in those cases). If all goes well, the computation will soon appear [16].

Taking the Schubert variety to be of a special kind and the point to be the "identity coset," our problem specializes to a problem about Pfaffian ideals considered in [5, 2]. On the other side of the spectrum from the identity coset, so to speak, lie the "generic singularities," points that are generic in the complement of the open orbit of the stabiliser of the Schubert variety. For these, a geometric solution to the problem appears in [1].

Given that our solution of the problem is but a translation, it makes sense to ask if one can extract more tangible information—closed form formulas for example—from our alternative description. See the papers quoted in the previous paragraph and also [3] for some answers in the special cases they consider.

Organization of the paper

The table of contents indicates how the paper is organized. There is a brief description at the beginning of every subdivision of the contents therein. An index of definitions and notation is included, for it would otherwise be difficult to find the meanings of certain words and symbols.

Important note added

The recent article [6] treats some of the questions addressed here and some that could be addressed by using the main result proved here. It includes:

- an interpretation of the multiplicity similar to ours.
- a closed formula for the multiplicity (as a specialization of a factorial Schur function), thereby answering the question we raised above.
- a formula for the restriction to the torus fixed point of the equivariant cohomology class of a Schubert variety.

The approach in [6] is quite different from ours. In fact, it is the opposite of ours in that it circumvents the lack of results about initial ideals of tangent cones, while our prime motivation is to remedy the lack. The starting points in the two approaches are also different: [6] takes off from certain results of Kostant-Kumar and Arabia on equivariant cohomology, while our launchpad is standard monomial theory.

The appearance of [6] notwithstanding, our approach is worthwhile, for, quite apart from the difference in starting points, there is no way, as far as we can tell, to the Hilbert function via the approach of [6], nor to the initial ideal, both of which are interesting in their own right.

Part I
The theorem

Definitions are recalled, the problem formulated, and the theorem stated.

1 The set up

In this section, we state the problem to be addressed after recalling the necessary basic definitions, make some choices that are convenient for studying the problem, and see why it is enough to focus on a particular case of the problem.

1.1 The statement of the problem

Fix an algebraically closed field of characteristic not equal to 2. Fix a vector space V of finite dimension n over this field and a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ on V. Let d be the integer such that either $n = 2d$ or $n = 2d + 1$. A linear subspace of V is said to be *isotropic* if the form $\langle\ ,\ \rangle$ vanishes identically on it. It is elementary to see that an isotropic subspace of V has dimension at most d and that every isotropic subspace is contained in one of dimension d. Denote by $\mathfrak{M}_d(V)'$ the closed sub-variety of the Grassmannian of d-dimensional subspaces consisting of the points corresponding to isotropic subspaces.

The orthogonal group $\mathrm{O}(V)$ of linear automorphisms of V preserving $\langle\ ,\ \rangle$ acts transitively on $\mathfrak{M}_d(V)'$, for by Witt's theorem an isometry between subspaces can be lifted to one of the whole vector space. If n is odd the special orthogonal group $\mathrm{SO}(V)$ (consisting of form preserving linear automorphisms with trivial determinant) itself acts transitively on $\mathfrak{M}_d(V)'$. If n is even the special orthogonal group $\mathrm{SO}(V)$ does not act transitively on $\mathfrak{M}_d(V)'$, and $\mathfrak{M}_d(V)'$ has two connected components. We define the *orthogonal Grassmannian* $\mathfrak{M}_d(V)$ to be $\mathfrak{M}_d(V)'$ if n is odd and to be one of the two components of $\mathfrak{M}_d(V)'$ if n is even.

The *Schubert varieties* of $\mathfrak{M}_d(V)$ are defined to be the B-orbit closures in $\mathfrak{M}_d(V)$ (with canonical reduced scheme structure), where B is a Borel subgroup of $\mathrm{SO}(V)$. The choice of B is immaterial, for any two of them are conjugate. The question that is tackled in this paper is this: given a point on a Schubert variety in $\mathfrak{M}_d(V)$, how to compute the multiplicity (and more generally, the Hilbert function) of the Schubert variety at the given point? The answers are contained in Theorem 2.3.1 and Corollary 2.3.2. But in order to make sense of those statements, we need some preparation.

1.2 Some convenient choices

We now make some choices that are convenient for the study of Schubert varieties. For k an integer such that $1 \leq k \leq n$, set $k^* := n + 1 - k$. Fix a basis

$e_1, \ \ldots, \ e_n$ of V such that

$$\langle e_i, e_k \rangle = \begin{cases} 1 & \text{if } i = k^* \\ 0 & \text{otherwise} \end{cases}$$

The advantage of this choice is: the elements of $\mathrm{SO}(V)$ for which each e_k is an eigenvector form a maximal torus, and the elements that are upper triangular with respect to this basis form a Borel subgroup (a linear transformation is *upper triangular* if for each k, $1 \leq k \leq n$, the image of e_k under the transformation is a linear combination of $e_1, \ldots, e_k$). We denote this maximal torus and this Borel subgroup by T and B respectively. Our Schubert varieties will be orbit closures of this particular Borel subgroup B.

The B-orbits of $\mathfrak{M}_d(V)'$ are naturally indexed by its T-fixed points: each orbit contains one and only one such point. The T-fixed points are evidently of the form $\langle e_{i_1}, \ldots, e_{i_d} \rangle$, where $1 \leq i_1 < \ldots < i_d \leq n$ and for each k, $1 \leq k \leq d$, there does not exist j, $1 \leq j \leq d$, such that $i_k^* = i_j$—in other words, for each ℓ, $1 \leq \ell \leq n$, such that $\ell \neq \ell^*$, exactly one of ℓ and ℓ^* appears in $\{i_1, \ldots, i_d\}$; in addition, if n is odd, then $d+1$ does not appear in $\{i_1, \ldots, i_d\}$. Denote the set of such d-element subsets $\{i_1 < \ldots < i_d\}$ by I'_n. We thus have a bijective correspondence between I'_n and the B-orbits of $\mathfrak{M}_d(V)'$. Each B-orbit being irreducible and open in its closure, it follows that B-orbit closures are indexed by the B-orbits. Thus I'_n is an indexing set for B-orbit closures in $\mathfrak{M}_d(V)'$.

Suppose that n is even—it will be shown presently that it is enough to consider this case. As already observed, $\mathfrak{M}_d(V)'$ has two connected components on each of which $\mathrm{SO}(V)$ acts transitively. The B-orbits belong to one or the other component accordingly as the parity of the cardinality of the number of entries bigger than d in the corresponding element of I'_n. We take $\mathfrak{M}_d(V)$ to be the component in which these cardinalities are even. We let I_n denote the subset of I'_n consisting of elements for which this cardinality is even. Schubert varieties in $\mathfrak{M}_d(V)$ are thus indexed by elements of I_n.

1.3 Reduction to the case n even

We now argue that it is enough to consider the case n even. Suppose that n is odd. Let $\widetilde{n} := n+1$ and $\widetilde{V}$ be a vector space of dimension $\widetilde{n}$ with a non-degenerate symmetric form. Let $\widetilde{e}_1, \ldots, \widetilde{e}_{\widetilde{n}}$ be a basis of $\widetilde{V}$ as in 1.2. Put $e := \widetilde{e}_{d+1}$ and $f := \widetilde{e}_{d+2}$. Take λ to be an element of the field such that $\lambda^2 = 1/2$. We can take V to be the subspace of $\widetilde{V}$ spanned by the vectors $\widetilde{e}_1, \ldots, \widetilde{e}_d, \lambda e + \lambda f, \widetilde{e}_{d+3}, \ldots, \widetilde{e}_{\widetilde{n}}$, and a basis of V to be these vectors in that order.

There is a natural map from $\mathfrak{M}_{d+1}(\widetilde{V})'$ to $\mathfrak{M}_d(V)$: intersecting with V an isotropic subspace of $\widetilde{V}$ of dimension $d+1$ gives an isotropic subspace of V of dimension d. This map is onto, for every isotropic subspace of $\widetilde{V}$ (and hence of V) is contained in an isotropic subspace of $\widetilde{V}$ of dimension $d+1$. It is also elementary to see that the map is two-to-one (essentially because in a two-dimensional space with a non-degenerate symmetric form there are two isotropic

lines), and that the two points in any fiber lie one in each component (there is clearly an element in $\mathrm{O}(\widetilde{V}) \setminus \mathrm{SO}(\widetilde{V})$ that moves one element of the fiber to the other, and so if there was an element of $\mathrm{SO}(\widetilde{V})$ that also moved one point to the other, the isotropy at the point would not be contained in $\mathrm{SO}(\widetilde{V})$, a contradiction).

We therefore get a natural isomorphism between $\mathfrak{M}_{d+1}(\widetilde{V})$ and $\mathfrak{M}_d(V)$. We will now show that the $\widetilde{B}$-orbits in $\mathfrak{M}_{d+1}(\widetilde{V})$ correspond under the isomorphism to B-orbits of $\mathfrak{M}_d(V)$ (we denote by $\widetilde{T}$ and $\widetilde{B}$ the maximal torus and Borel subgroups of $\mathrm{SO}(\widetilde{V})$ as in §1.2). It will then follow that Schubert varieties in $\mathfrak{M}_{d+1}(\widetilde{V})$ are isomorphic to those in $\mathfrak{M}_d(V)$ and the purpose of this subsection will be achieved.

The group $\mathrm{SO}(V)$ can be realized as the subgroup of $\mathrm{SO}(\widetilde{V})$ consisting of the elements that fix $e - f$. The isomorphism $\mathfrak{M}_{d+1}(\widetilde{V}) \cong \mathfrak{M}_d(V)$ above is equivariant for $\mathrm{SO}(V)$, and we have $\widetilde{T} \cap \mathrm{SO}(V) = T$ and $\widetilde{B} \cap \mathrm{SO}(V) = B$. It should now be clear that the preimages in $\mathfrak{M}_{d+1}(\widetilde{V})$ of two elements in the same B-orbit of $\mathfrak{M}_d(V)$ are in the same $\widetilde{B}$-orbit: an element of B that moves one to the other considered as an element of $\widetilde{B}$ moves also the preimage of the one to that of the other.

On the other hand, the preimages of distinct T-fixed points are distinct $\widetilde{T}$-fixed points, the corresponding map from I'_n to $I_{\widetilde{n}}$ being given as follows:

$$\underline{i} = \{i_1 < \ldots < i_d\} \mapsto \begin{cases} \{\widetilde{i}_1, \ldots, \widetilde{i}_d, d+1\} & \text{if } \underline{i} \in I_n \\ \{\widetilde{i}_1, \ldots, \widetilde{i}_d, d+2\} & \text{if } \underline{i} \in I'_n \setminus I_n \end{cases}$$

where

$$\widetilde{i}_k = \begin{cases} i_k & \text{if } i_k \leq d \\ i_k + 1 & \text{if } i_k \geq d+2 \end{cases}$$

(Note that $d+1$ never occurs as an entry in any element of I'_n and that the elements $\widetilde{i}_1, \ldots, \widetilde{i}_d, d+1$ (respectively $\widetilde{i}_1, \ldots, \widetilde{i}_d, d+2$) are not in increasing order except in the trivial case $\underline{i} = \{1 < \ldots < d\}$.) Given that each B-orbit has a T-fixed point and that distinct $\widetilde{T}$-fixed points belong to distinct $\widetilde{B}$-orbits, this implies that the preimages of two elements in distinct B-orbits belong to distinct $\widetilde{B}$-orbits, and the proof is over. $\square$

2 The theorem

The purpose of this section is to state the main theorem and its corollary. We first set down some basic notation and two fundamental definitions needed in order to state the theorem.

2.1 Basic notation

We keep the terminology and notations of §1.1, 1.2. As observed in §1.3, it is enough to consider the case n even. So from now on let $n = 2d$. Recall that,

for an integer k, $1 \leq k \leq 2d$, $k^* := 2d + 1 - k$. As observed in §1.2, Schubert varieties in $\mathfrak{M}_d(V)$ are indexed by I_n.

Since d now determines n, we will henceforth write $I(d)$ instead of I_n. In other words, $I(d)$ is the set of d-element of subsets of $\{1, \ldots, 2d\}$ such that

- for each k, $1 \leq k \leq 2d$, the subset contains exactly one of k, k^*, and
- the number of elements in the subset that exceed d is even.

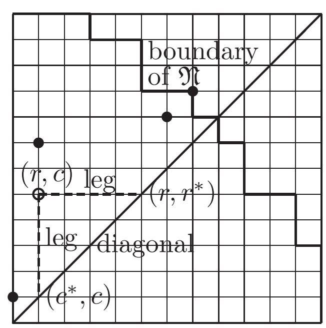
We write $I(d, 2d)$ for the set of all d-element subsets of $\{1, \ldots, 2d\}$. There is a natural partial order on $I(d, 2d)$ and so also on $I(d)$: $v = (v_1 < \ldots < v_d) \leq w = (w_1 < \ldots < w_d)$ if and only if $v_1 \leq w_1, \ldots, v_d \leq w_d$.

Given $v \in I(d)$, the corresponding T-fixed point in $\mathfrak{M}_d(V)$ (namely, the span of $e_{v_1}, \ldots, e_{v_d}$) is denoted $\mathfrak{e}^v$. Given $w \in I(d)$, the corresponding Schubert variety in $\mathfrak{M}_d(V)$ (which, by definition, is the closure of the B-orbit of the T-fixed point $\mathfrak{e}^w$ with canonical reduced scheme structure) is denoted $X(w)$. The point $\mathfrak{e}^v$ belongs to $X(w)$ if and only if $v \leq w$ in the partial order just defined. Since, under the natural action of B on $X(w)$, each point of $X(w)$ is in the B-orbit of a T-fixed point $\mathfrak{e}^v$ for some v such that $v \leq w$, it is enough to focus attention on such T-fixed points.

For the rest of this section an element v of $I(d)$ will remain fixed.

We will be dealing extensively with ordered pairs (r, c), $1 \leq r, c \leq 2d$, such that r is not and c is an entry of v. Let $\mathfrak{R}$ denote the set of all such ordered pairs, and set

$$\begin{aligned}
\mathfrak{N} &:= \{(r,c) \in \mathfrak{R} \mid r > c\} \\
\mathfrak{O}\mathfrak{R} &:= \{(r,c) \in \mathfrak{R} \mid r < c^*\} \\
\mathfrak{O}\mathfrak{N} &:= \{(r,c) \in \mathfrak{R} \mid r > c, r < c^*\} \\
&= \mathfrak{O}\mathfrak{R} \cap \mathfrak{N} \\
\mathfrak{d} &:= \{(r,c) \in \mathfrak{R} \mid r = c^*\}
\end{aligned}$$



The picture shows a drawing of $\mathfrak{R}$. We think of r and c in (r, c) as row index and column index respectively. The columns are indexed from left to right by the entries of v in ascending order, the rows from top to bottom by the entries of $\{1, \ldots, 2d\} \setminus v$ in ascending order. The points of $\mathfrak{d}$ are those on the diagonal, the points of $\mathfrak{O}\mathfrak{R}$ are those that are (strictly) above the diagonal, and the points of $\mathfrak{N}$ are those that are to the South-West of the polyline captioned "boundary of $\mathfrak{N}$"—we draw the boundary so that points on the boundary belong to $\mathfrak{N}$. The reader can readily verify that $d = 13$ and $v = (1, 2, 3, 4, 6, 7, 10, 11, 13, 15, 18, 19, 22)$ for the particular picture drawn. The points of $\mathfrak{O}\mathfrak{N}$ indicated by solid circles form a v-chain (see §2.2.1 below).

We will be considering *monomials*, also called *multisets*, in some of these sets. A *monomial*, as usual, is a subset with each member being allowed a multiplicity (taking values in the non-negative integers). The *degree* of a monomial has also

the usual sense: it is the sum of the multiplicities in the monomial over all elements of the set. The *intersection* of a monomial in a set with a subset of the set has also the natural meaning: it is a monomial in the subset, the multiplicities being those in the original monomial.

We will refer to $\mathfrak{d}$ as the *diagonal*.

2.2 Two fundamental definitions

2.2.1 Definition of v-chain

Given two elements (R, C) and (r, c) in $\mathfrak{O}\mathfrak{N}$, we write $(R, C) > (r, c)$ if $R > r$ and $C < c$ (note that these are strict inequalities). An ordered sequence $\alpha, \beta, \ldots$ of elements of $\mathfrak{O}\mathfrak{N}$ is called a *v-chain* if $\alpha > \beta > \ldots$. A v-chain $\alpha_1 > \ldots > \alpha_\ell$ has *head* α_1, *tail* α_ℓ, and *length* ℓ.

2.2.2 Definition of $\mathfrak{O}$-domination

To a v-chain $C : \alpha_1 > \alpha_2 > \ldots$ in $\mathfrak{O}\mathfrak{N}$ there corresponds, as described in §5.3.3, a subset $\mathfrak{S}_C$ of $\mathfrak{N}$ which, as observed in Proposition 5.3.5, is "distinguished" in the sense of §5.1.1. To a distinguished subset of $\mathfrak{N}$ there corresponds, as described below in §5.1.2, an element of $I(d, 2d)$. Following these correspondences through, we get an element of $I(d, 2d)$ attached to the v-chain C. Let $w(C)$ denote this element—sometimes we write w_C. (All this makes sense even when C is empty—$w(C)$ will turn out to be v itself in that case.)

Furthermore, as will be obvious from its definition, the monomial $\mathfrak{S}_C$ is "symmetric" in the sense of §5.2.2 and contains evenly many elements of the diagonal $\mathfrak{d}$. Thus, by Proposition 5.2.1, the element $w(C)$ of $I(d, 2d)$ belongs to $I(d)$.

An element w of $I(d)$ is said to *$\mathfrak{O}$-dominate* C if $w \geq w(C)$, or, equivalently—and this is important for the proofs—if w dominates in the sense of [7] the monomial $\mathfrak{S}_C$ (for the proof of the equivalence, see [7, Lemma 5.5]). An element w of $I(d)$ *$\mathfrak{O}$-dominates* a monomial $\mathfrak{S}$ of $\mathfrak{O}\mathfrak{N}$ (repsectively of $\mathfrak{O}\mathfrak{R}$) if it $\mathfrak{O}$-dominates every v-chain in $\mathfrak{S}$ (respectively in $\mathfrak{S} \cap \mathfrak{O}\mathfrak{N}$).

2.3 The main theorem and its corollary

Theorem 2.3.1 *Fix a positive integer d and elements $v \leq w$ of $I(d)$. Let V be a vector space of dimension $2d$ with a symmetric non-degenerate bilinear form (over a field of characteristic not 2). Let $X(w)$ be the Schubert variety corresponding to w in the orthogonal Grassmannian $\mathfrak{M}_d(V)$, and $\mathfrak{e}^v$ the torus fixed point of $X(w)$ corresponding to v. Let R_v^w denote the associated graded ring with respect to the unique maximal ideal of the local ring of germs at $\mathfrak{e}^v$ of functions on $X(w)$. Then, for any non-negative integer m, the dimension as a vector space of the homogeneous piece of R_v^w of degree m equals the cardinality of the set $S^w(v)(m)$ of monomials of degree m of $\mathfrak{O}\mathfrak{R}$ that are $\mathfrak{O}$-dominated by w.*

The proof of this theorem occupies us for most of this paper. It is reduced in §3, by an application of standard monomial theory, to combinatorics. The resulting combinatorial problem is solved in §4–10. For now, let us note the following immediate consequence:

Corollary 2.3.2 *The multiplicity at the point $\mathfrak{e}^v$ of the Schubert variety $X(w)$ equals the number of monomials in $\mathfrak{O}\mathfrak{N}$ of maximal cardinality that are square-free and $\mathfrak{O}$-dominated by w.*

PROOF: The proof of Corollary 2.2 of [7] holds verbatim here too. $\square$

Part II

From geometry to combinatorics

The problem is translated from geometry to combinatorics. The main combinatorial results are formulated.

3 Reduction to combinatorics

In this section we translate the problem from geometry to combinatorics. In §3.1 we recall from [17] the theorem that enables the translation. The translation itself is done in 3.2 and follows [14].

3.1 Homogeneous co-ordinate ring of the Schubert variety $X(w)$

3.1.1 The line bundle L on $\mathfrak{M}_d(V)$

Let $\mathfrak{M}_d(V) \subseteq G_d(V) \hookrightarrow \mathbb{P}(\wedge^d V)$ be the Plücker embedding (where $G_d(V)$ denotes the Grassmannian of all d-dimensional subspaces of V). The pull-back to $\mathfrak{M}_d(V)$ of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\wedge^d V)$ is the square of the ample generator of the Picard group of $\mathfrak{M}_d(V)$. Letting L denote the ample generator, we observe that it is very ample and want to describe the homogeneous coordinate rings of $\mathfrak{M}_d(V)$ and its Schubert subvarieties in the embedding defined by L.

3.1.2 The section q_θ of L

For θ in $I(d, 2d)$, let p_θ denote the corresponding Plücker coordinate. Consider the affine patch $\mathbb{A}$ of $\mathbb{P}(\wedge^d V)$ given by $p_\epsilon = 1$, where $\epsilon := (1, \ldots, d)$. The intersection $\mathbb{A} \cap G_d(V)$ of this patch with the Grassmannian is an affine space. Indeed the d-plane corresponding to an arbitrary point z of $\mathbb{A} \cap G_d(V)$ has a basis consisting of column vectors of a matrix of the form

$$M = \begin{pmatrix} I \\ A \end{pmatrix}$$

where I is the identity matrix and A an arbitrary matrix both of size $d \times d$. The association $z \mapsto A$ is bijective. The restriction of a Plücker coordinate p_θ to $\mathbb{A} \cap G_d(V)$ is given by the determinant of a submatrix of size $d \times d$ of M, the entries of θ determining the rows to be chosen from M to form the submatrix.

As can be readily verified, a point z of $\mathbb{A} \cap G_d(V)$ represents an isotropic subspace if and only if the corresponding matrix $A = (a_{ij})$ is *skew-symmetric with respect to the anti-diagonal*: $a_{ij} + a_{j^*i^*} = 0$, where the columns and rows of A are numbered $1, \ldots, d$ and $d+1, \ldots, 2d$ respectively. For example, if $d = 4$, then a matrix that is skew-symmetric with respect to the anti-diagonal looks

like this:

$$\begin{pmatrix} -d & -c & -b & 0 \\ -g & -f & 0 & b \\ -i & 0 & f & c \\ 0 & i & g & d \end{pmatrix}$$

Since the set of these matrices is connected and contains the point that is spanned by $e_1, \ldots, e_d$, it follows that $\mathbb{A} \cap G_d(V)$ does not intersect the other component of $\mathfrak{M}_d(V)'$. In other words, p_ϵ vanishes everywhere on $\mathfrak{M}_d(V)' \setminus \mathfrak{M}_d(V)$.

Now suppose that θ belongs to $I(d)$. Computing p_θ / p_ϵ as a function on the affine patch $p_\epsilon \neq 0$, we see that it is the determinant of a skew-symmetric matrix of even size, and therefore a square. The square root, which is determined up to sign, is called the *Pfaffian*. This suggests that p_θ itself is a square: more precisely that there exists a section q_θ of the line bundle L on $\mathfrak{M}_d(V)$ such that $q_\theta^2 = p_\theta$. A weight calculation confirms this to be the case. The q_θ are also called *Pfaffians*.

3.1.3 Standard monomial theory for $\mathfrak{M}_d(V)$

A *standard monomial* in $I(d)$ is a totally ordered sequence $\theta_1 \geq \ldots \geq \theta_t$ (with repetitions allowed) of elements of $I(d)$. Such a standard monomial is said to be *w-dominated* for $w \in I(d)$ if $w \geq \theta_1$. To a standard monomial $\theta_1 \geq \ldots \geq \theta_t$ in $I(d)$ we associate the product $q_{\theta_1} \cdots q_{\theta_t}$, where the q_θ are the sections defined above of the line bundle L. Such a product is also called a *standard monomial* and it is said to be *dominated by w* for $w \in I(d)$ if the underlying monomial in $I(d)$ is dominated by w. *Standard monomial theory* for $\mathfrak{M}_d(V)$ says:

Theorem 3.1.1 (Seshadri [17]) *Standard monomials $q_{\theta_1} \cdots q_{\theta_r}$ of degree r form a basis for the space of forms of degree r in the homogeneous coordinate ring of $\mathfrak{M}_d(V)$ in the embedding defined by the ample generator L of the Picard group. More generally, for $w \in I(d)$, the w-dominated standard monomials of degree r form a basis for the space of forms of degree r in the homogeneous coordinate ring of the Schubert subvariety $X(w)$ of $\mathfrak{M}_d(V)$.*

3.2 Co-ordinate rings of affine patches and tangent cones of $X(w)$

From Theorem 3.1.1 one can deduce rather easily, as we now show, bases for co-ordinate rings of affine patches of the form $q_v \neq 0$ and of tangent cones of Schubert varieties. An element v of $I(d)$ will remain fixed for the rest of this section. To simplify notation we will suppress explicit reference to v.

3.2.1 Standard monomial theory for affine patches

Let $\mathbb{A}$ denote the affine patch of $\mathbb{P}(H^0(\mathfrak{M}_d(V), L)^*)$ given by $q_v \neq 0$. The origin of the affine space $\mathbb{A}$ is identified as the T-fixed point $\mathfrak{e}^v$. The functions $f_\theta := q_\theta / q_v$, $v \neq \theta \in I(d)$, provide a set of coordinate functions on $\mathbb{A}$. Monomials

in these f_θ form a $\mathfrak{k}$-basis for the polynomial ring $\mathfrak{k}[\mathbb{A}]$ of functions on $\mathbb{A}$, where $\mathfrak{k}$ denotes the underlying field.

Fix $w \geq v$ in $I(d)$, so that the point $\mathfrak{e}^v$ belongs to the Schubert variety $X(w)$, and let $Y(w)$ be the affine patch of $X(w)$ defined thus:

$$Y(w) := X(w) \cap \mathbb{A}.$$

The coordinate ring $\mathfrak{k}[Y(w)]$ of $Y(w)$ is a quotient of the polynomial ring $\mathfrak{k}[\mathbb{A}]$, and the proposition that follows identifies a subset of the monomials in f_θ which forms a $\mathfrak{k}$-basis for $\mathfrak{k}[Y(w)]$.

We say that a standard monomial $\theta_1 \geq \ldots \geq \theta_t$ in $I(d)$ is *v-compatible* if for each k, $1 \leq k \leq t$, either $\theta_k \gneq v$ or $v \gneq \theta_k$. Given w in $I(d)$, we denote by SM^w the set of w-dominated v-compatible standard monomials.

Proposition 3.2.1 *As $\theta_1 \geq \ldots \geq \theta_t$ runs over the set SM^w of w-dominated v-compatible standard monomials, the elements $f_{\theta_1} \cdots f_{\theta_t}$ form a basis for the coordinate ring $\mathfrak{k}[Y(w)]$ of the affine patch $Y(w) = X(w) \cap \mathbb{A}$ of the Schubert variety $X(w)$.*

Proof: The proof is similar to the proof of Proposition 3.1 of [7]. First consider a linear dependence relation among the $f_{\theta_1} \cdots f_{\theta_t}$. Replacing f_θ by q_θ and "homogenizing" by q_v yields a linear dependence relation among the w-dominated standard monomials $q_{\theta_1} \cdots q_{\theta_s}$ restricted to $X(w)$, and so the original relation must only have been the trivial one, for by Theorem 3.1.1 the $q_{\theta_1} \cdots q_{\theta_s}$ are linearly independent on $X(w)$.

To prove that $f_{\theta_1} \cdots f_{\theta_t}$ generate $\mathfrak{k}[Y(w)]$ as a vector space, we make the following claim: if $q_{\mu_1} \cdots q_{\mu_r}$ be any monomial in the Pfaffians q_θ, and $q_{\tau_1} \cdots q_{\tau_s}$ a standard monomial that occurs with non-zero co-efficient in the expression for (the restriction to $X(w)$ of) $q_{\mu_1} \cdots q_{\mu_r}$ as a linear combination of w-dominated standard monomials, then $\tau_1 \cup \cdots \cup \tau_s = \mu_1 \cup \cdots \cup \mu_r$ as multisets of $\{1, \ldots, 2d\}$. To prove the claim, consider the maximal torus T of $\mathrm{SO}(V)$ as in §1.2. The affine patch $\mathbb{A}$ is T-stable and there is an action of T on $\mathfrak{k}[Y(w)]$. The sections q_θ are eigenvectors for T with corresponding characters $\epsilon_{\theta_1} + \cdots + \epsilon_{\theta_d}$, where ϵ_k denotes the character of T given by the projection to the diagonal entry on row k. The claim now follows since eigenvectors corresponding to different characters are linearly independent.

Let $f_{\mu_1} \cdots f_{\mu_r}$ be an arbitrary monomial in the f_θ. Fix an integer h such that $h > r(d-1)$ and consider the expression for (the restriction to $X(w)$ of) $q_{\mu_1} \cdots q_{\mu_r} \cdot q_v^h$ as a linear combination of w-dominated standard monomials. We claim that q_v occurs in every standard monomial $q_{\tau_1} \cdots q_{\tau_{r+h}}$ in this expression (from which it will follow that the τ_j are all comparable to v). Suppose that none of $\tau_1, \ldots, \tau_{r+h}$ equals v. For each τ_j there is at least one entry of v that does not occur in it. The number of occurrences of entries of v in $\tau_1 \cup \cdots \cup \tau_{r+h}$ is thus at most $(r+h)(d-1)$. But these entries occur at least hd times in $\mu_1 \cup \cdots \cup \mu_r \cup v \cup \cdots \cup v$ (where v is repeated h times), a contradiction to the claim proved in the previous paragraph. Hence our claim is proved. Dividing by q_v^{r+h} the expression for $q_{\mu_1} \cdots q_{\mu_r} . q_v^h$ as a linear combination of w-dominated

standard monomials provides an expression for $f_{\mu_1}\cdots f_{\mu_r}$ as a linear combination of $f_{\theta_1}\cdots f_{\theta_t}$, as $\theta_1 \geq \ldots \geq \theta_t$ varies over SM^w. $\square$

3.2.2 Standard monomial theory for tangent cones

The affine patch $\mathfrak{M}_d(V)\cap\mathbb{A}$ of the orthogonal Grassmannian $\mathfrak{M}_d(V)$ is an affine space whose coordinate ring can be taken to be the polynomial ring in variables of the form $X_{(r,c)}$ with $(r,c)\in\mathfrak{OR}$, where (as in §2.1)

$$\mathfrak{OR} = \{(r,c)\,|\,1\leq r,c\leq 2d, r\notin v, c\in v, r<c^*\}$$

Taking $d=5$ and $v=(1,3,4,6,9)$ for example, a general element of $\mathfrak{M}_d(V)\cap\mathbb{A}$ has a basis consisting of column vectors of a matrix of the following form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ X_{21} & X_{23} & X_{24} & X_{26} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ X_{51} & X_{53} & X_{54} & 0 & -X_{26} \\ 0 & 0 & 0 & 1 & 0 \\ X_{71} & X_{73} & 0 & -X_{54} & -X_{24} \\ X_{81} & 0 & -X_{73} & -X_{53} & -X_{23} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -X_{81} & -X_{71} & -X_{51} & -X_{21} \end{pmatrix}$$

The expression for $f_\theta = q_\theta/q_v$ in terms of the $X_{(r,c)}$ is a square root of the determinant of the submatrix of a matrix like the one above obtained by choosing the rows given by the entries of θ. Thus f_θ is a homogeneous polynomial of degree the v-degree of θ, where the *v-degree* of θ is defined as one half of the cardinality of $v\setminus\theta$.

Since the ideal of the Schubert variety $X(w)$ in the homogeneous coordinate ring of $\mathfrak{M}_d(V)$ is generated[2] by the q_τ, $\tau\in I(d)$ such that $\tau\not\leq w$, it follows that the ideal of $Y(w) := X(w)\cap\mathbb{A}$ in $\mathfrak{M}_d(V)\cap\mathbb{A}$ is generated by the the f_τ, $\tau\in I(d)$ such that $\tau\not\leq w$. We are interested in the tangent cone to $X(w)$ at $\mathfrak{e}^v$ (or, what is the same, the tangent cone to $Y(w)$ at the origin), and since $\mathfrak{k}[Y(w)]$ is graded, its associated graded ring with respect to the maximal ideal corresponding to the origin is $\mathfrak{k}[Y(w)]$ itself.

Proposition 3.2.1 says that the graded piece of $\mathfrak{k}[Y(w)]$ of degree m is generated as a $\mathfrak{k}$-vector space by elements of degree m of the set SM^w of w-dominated v-compatible standard monomials, where the *degree* of a standard monomial $\theta_1\geq\ldots\geq\theta_t$ is defined to be the sum of the v-degrees of $\theta_1,\ldots,\theta_t$. To prove Theorem 2.3.1 it therefore suffices to prove the following:

[2]This is a consequence of Theorem 3.1.1. It is easy to see that the q_τ such that $\tau\not\leq w$ vanish on $X(w)$. Since all standard monomials form a basis for the homogeneous coordinate ring of $\mathfrak{M}_d(V)$ in $\mathbb{P}(H^0(\mathfrak{M}_d(V),L)^*)$, it follows that w-dominated standard monomials span the quotient ring by the ideal generated by such q_τ. Since such monomials are linearly independent in the homogeneous coordinate ring of $X(w)$, the desired result follows.

Theorem 3.2.2 *The set $SM^w(m)$ of standard monomials in $I(d)$ of degree m that are w-dominated and v-compatible is in bijection with the set $S^w(v)(m)$ of monomials in $\mathfrak{O}\mathfrak{R}$ of degree m that are $\mathfrak{O}$-dominated by w.*

4 Further reductions

In the last section, we reduced the proof of our main theorem (Theorem 2.3.1) to that of Theorem 3.2.2. We now reduce the proof of Theorem 3.2.2 to that of Propositions 4.1.1, 4.1.2 and 4.1.3 below. These propositions will eventually be proved in §10.

4.1 The main propositions

Fix once and for all an element v of $I(d)$. The bijection stated in Theorem 3.2.2 will be described by means of two maps $\mathfrak{O}\pi$ and $\mathfrak{O}\phi$ whose definitions will be given in §7 and §8 below. We will now state some properties of these maps. In §4.2 we will see how Theorem 3.2.2 follows once these properties are established.

The map $\mathfrak{O}\pi$ associates to a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$ a pair $(w, \mathfrak{S}')$ consisting of an element w of $I(d)$ and a "smaller" monomial $\mathfrak{S}'$ in $\mathfrak{O}\mathfrak{N}$. This map enjoys the following good properties:

Proposition 4.1.1

1. *$w \geq v$.*
2. *v-degree(w) + degree$(\mathfrak{S}')$ = degree$(\mathfrak{S})$.*
3. *w $\mathfrak{O}$-dominates $\mathfrak{S}'$.*
4. *w is the least element of $I(d)$ that $\mathfrak{O}$-dominates $\mathfrak{S}$.*

The map $\mathfrak{O}\phi$, on the other hand, associates a monomial in $\mathfrak{O}\mathfrak{N}$ to a pair $(w, \mathfrak{T})$ consisting of an element w of $I(d)$ with $w \geq v$ and a monomial $\mathfrak{T}$ in $\mathfrak{O}\mathfrak{N}$ that is $\mathfrak{O}$-dominated by w.

Proposition 4.1.2 *The maps $\mathfrak{O}\pi$ and $\mathfrak{O}\phi$ are inverses of each other.*

For an integer f, $1 \leq f \leq 2d$, consider the following conditions, the first on a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$, the second on an element w of $I(d)$:

> (‡) f is not the row index of any element of $\mathfrak{S}$ and $f^\star$ is not the column index of any element of $\mathfrak{S}$.
> (‡) f is not an entry of w.

(It is convenient to the use the same notation (‡) for both conditions.)

Proposition 4.1.3 *Assume that v satisfies (‡)—all references to (‡) in this proposition are with respect to a fixed f, $1 \leq f \leq 2d$.*

1. *Let w be an element of $I(d)$ with $w \geq v$ and $\mathfrak{T}$ a monomial in $\mathfrak{O}\mathfrak{N}$ that is $\mathfrak{O}$-dominated by w. If w and $\mathfrak{T}$ both satisfy (‡), then so does $\mathfrak{O}\phi(w, \mathfrak{T})$.*

2. *If a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$ satisfies (‡), then so do the "components" w and $\mathfrak{S}'$ of its image under $\mathfrak{O}\pi$.*

4.2 From the main propositions to the main theorem

Let us now see how Theorem 3.2.2 follows from the propositions of §4.1. Most of the following argument runs parallel to its counterparts in the case of the Grassmannian and symplectic Grassmannian (Propositions 4.1.1 and 4.1.2 have their counterparts in [7, 4]), but, in the case that d is odd, the part involving the "mirror image" requires additional work. This is where Proposition 4.1.3 comes in.

Let S, T, and U, denote respectively the sets of monomials in $\mathfrak{O}\mathfrak{R}$, $\mathfrak{O}\mathfrak{N}$, and $\mathfrak{O}\mathfrak{R}\setminus\mathfrak{O}\mathfrak{N}$. Let SM_v denote the set of v-compatible standard monomials that are "anti-dominated" by v: a standard monomial $\theta_1 \geq \ldots \geq \theta_t$ is *anti-dominated* by v if $\theta_t \geq v$ (we can also write $\theta_t > v$ since $\theta_t \neq v$ by v-compatibility).

Define the *domination map* from T to $I(d)$ by sending a monomial in $\mathfrak{O}\mathfrak{N}$ to the least element that $\mathfrak{O}$-dominates it. Define the *domination map* from SM_v to $I(d)$ by sending $\theta_1 \geq \ldots \geq \theta_t$ to θ_1. Both these maps take, by definition, the value v on the empty monomial.

Notation 4.2.1 In the following, we use subscripts, superscripts, suffixes, and combinations thereof to modify the meanings of S, T, U, SM, and SM_v.

- superscript: this will be an element w of $I(d)$; when used on T it denotes $\mathfrak{O}$-domination (more precisely, T^w denotes the subset of T consisting of those elements that are $\mathfrak{O}$-dominated by w); when used on SM or SM_v it denotes domination by w.
- subscript: denotes anti-domination (applied only to standard monomials).
- suffix "(m)": indicates degree (for example, $SM_v^w(m)$ denotes the set of v-compatible standard monomials that are anti-dominated by v, dominated by w, and of degree m).

Repeated application of $\mathfrak{O}\pi$ gives a map from T to SM_v that commutes with domination (as just defined) and preserves degree. Repeated application of $\mathfrak{O}\phi$ gives a map from SM_v to T. These two maps being inverses of each other (Proposition 4.1.2) and so we have a bijection between SM_v and T. In fact, since domination and degree are respected (Proposition 4.1.1), we get a bijection $SM_v^w(m) \cong T^w(m)$.

As explained below, the "mirror image" of the bijection $SM_v(m) \cong T(m)$ gives a bijection $SM^v(m) \cong U(m)$. Putting these bijections together, we get the desired result:

$$\begin{aligned} SM^w(m) &= \bigcup_{k=0}^{m} SM_v^w(k) \times SM^v(m-k) \\ &\cong \bigcup_{k=0}^{m} T^w(k) \times U(m-k) = S^w(m). \end{aligned}$$

We now explain how to realize the bijection $SM^v(m) \cong U(m)$ as the "mirror image" of the bijection $SM_v(m) \cong T(m)$. For an element u of $I(d)$, define $u^* := (u_d^*, \ldots, u_1^*)$. In the case d is even, the association $u \mapsto u^*$ is an order reversing involution, and the argument in [4] for the symplectic Grassmannian holds here too. In the case d is odd, u^* is not an element of $I(d)$, and so some additional work is required.

Recall that a "base element" v of $I(d)$ has been fixed and that our notation does not explicitly indicate this dependence upon v: for example, $\mathfrak{OR}$ is dependent upon v. For a brief while now (until the end of this section) we need to simultaneously handle several base elements of $I(d)$. We will use the following convention: when the base element of $I(d)$ is not v, we will explicitly indicate it by means of a suffix. For instance, $SM(v^*)$ denotes the set of v^*-compatible standard monomials in $I(d)$.

Let us first do the case when d is even. We get a bijection $SM^v \cong SM_{v^*}(v^*)$ by associating to $\theta_1 \geq \ldots \geq \theta_t$ the element $\theta_t^* \geq \ldots \geq \theta_1^*$. The sum of the v-degrees of $\theta_1, \ldots, \theta_t$ equals the sum of the v^*-degrees of $\theta_t^*, \ldots, \theta_1^*$, so that we get a bijection $SM^v(m) \cong SM_{v^*}(v^*)(m)$.

For an element (r, c) of $\mathfrak{ON}(v^*)$, consider its flip (c, r). Since v belongs to $I(d)$, the complement of v^* in $\{1, \ldots, 2d\}$ is v, and it follows that (c, r) belongs to $\mathfrak{OR} \setminus \mathfrak{ON}$. This induces a degree preserving bijection $T(v^*) \cong U$. Putting this together with the bijection of the previous paragraph and the one deduced earlier in this section (using $\mathfrak{O}\pi$ and $\mathfrak{O}\phi$), we get what we want:

$$SM^v(m) \cong SM_{v^*}(v^*)(m) \cong T(v^*)(m) \cong U(m).$$

Now suppose that d is odd. Then the map $x \mapsto x^*$ does not map $I(d)$ to $I(d)$ but to $I(d)^*$ (defined as the set consisting of those elements u of $I(d, 2d)$ such that, for each k, $1 \leq k \leq 2d$, exactly one of k, k^* belongs to u, and the number of entries of u greater than d is odd). We define a map $u \mapsto \widetilde{u}$ from $I(d)^*$ to $I(d+1)$ as follows: $\widetilde{u} := \{\widetilde{u_1}, \ldots, \widetilde{u_d}, d+2\}$ (the elements are not in increasing order except in the trivial case $u = (1, \ldots, d)$), where, for an integer e, $1 \leq e \leq 2d$, we set

$$\widetilde{e} := \begin{cases} e & \text{if } 1 \leq e \leq d \\ e+2 & \text{if } d+1 \leq e \leq 2d \end{cases}$$

This map $u \mapsto \widetilde{u}$ is an order preserving injection.

Consider the composition $x \mapsto x^* \mapsto \widetilde{x^*}$ from $I(d)$ to $I(d+1)$. This is an order reversing injection. The induced map on standard monomials is an injection from SM^v to $SM_{\widetilde{v^*}}(\widetilde{v^*})$. It is readily seen that the image under this map is the subset $SM_{\widetilde{v^*}}(\widetilde{v^*})(\ddagger)$ consisting of those standard monomials all of whose elements satisfy ($\ddagger$) with $f = d+1$. We have already established (using the maps $\mathfrak{O}\pi$ and $\mathfrak{O}\phi$) a bijection $SM_{\widetilde{v^*}}(\widetilde{v^*}) \cong T(\widetilde{v^*})$. It follows from Proposition 4.1.3 that under this bijection the subset $SM_{\widetilde{v^*}}(\widetilde{v^*})(\ddagger)$ maps to $T(\widetilde{v^*})(\ddagger)$ (defined as the set of those monomials in $\mathfrak{ON}(\widetilde{v^*})$ satisfying ($\ddagger$) with $f = d+1$).

Now $T(\widetilde{v^*})(\ddagger)$ is in degree preserving bijection with U: every element of degree 1 of $T(\widetilde{v^*})(\ddagger)$ is uniquely of the form $(\widetilde{c}, \widetilde{r})$ for (r, c) in $\mathfrak{OR} \setminus \mathfrak{ON}$, and the

desired bijection is induced from this. Putting all of these together, we finally have

$$SM^v \cong SM_{\widetilde{v^*}}(\widetilde{v^*})(\ddagger) \cong T(\widetilde{v^*})(\ddagger) \cong U.$$

Thus, in order to prove our main theorem (Theorem 2.3.1), it suffices to describe the maps $\mathfrak{O}\pi$ and $\mathfrak{O}\phi$ and to prove Propositions 4.1.1–4.1.3.

Part III
The proof

The main combinatorial results formulated in §4.1 are proved. An attempt is made to maintain parallelism with the proofs in [7].

5 Terminology and notation

5.1 Distinguished subsets

5.1.1 Distinguished subsets of $\mathfrak{N}$

Following [7, §4], we define a multiset $\mathfrak{S}$ of $\mathfrak{N}$ to be *distinguished*, if, first of all, it is a subset in the usual sense (in other words, it is "multiplicity free"), and if, for any two distinct elements (R, C) and (r, c) of $\mathfrak{S}$, the following conditions are satisfied:

A. $R \neq r$ and $C \neq c$.

B. If $R > r$, then either $r < C$ or $C < c$.

In terms of pictures, condition A says that (r, c) cannot lie exactly due North or East of (R, C) (or the other way around); so we can assume, interchanging the two points if necessary, that (r, c) lies strictly to the Northeast or Northwest of (R, C); condition B now says that, if (r, c) lies to the Northwest of (R, C), then the point that is simultaneously due North of (R, C) and due East of (r, c) (namely (r, C)) does not belong to $\mathfrak{N}$.

5.1.2 Attaching elements of $I(d, 2d)$ to distinguished subsets of $\mathfrak{N}$

To a distinguished subset $\mathfrak{S}$ of $\mathfrak{N}$ there is naturally associated an element w of $I(d, 2d)$ as follows: start with v, remove all members of v which appear as column indices of elements of $\mathfrak{S}$, and add row indices of all elements of $\mathfrak{S}$. As observed in [7, Proposition 4.3], this association gives a bijection between distinguished subsets of $\mathfrak{N}$ and elements $w \geq v$ of $I(d, 2d)$. The unique distinguished subset of $\mathfrak{N}$ corresponding to an element $w \geq v$ of $I(d, 2d)$ is denoted $\mathfrak{S}_w$.

5.2 The involution

5.2.1 The involution # on $I(d, 2d)$

There are two natural order reversing involutions on $I(d, 2d)$. First there is $w \mapsto w^*$ induced by the natural order reversing involution $j \mapsto j^*$ on $\{1, \ldots, 2d\}$: here w^* has the obvious meaning, namely, it consists of all j^* such that j belongs to w. Then there is the map taking w to its complement $\{1, \ldots, 2d\} \setminus w$. These two involutions commute. Composing the two we get an order preserving involution on $I(d, 2d)$ which we denote by $w \mapsto w^{\#}$. The elements of the

subset $I(d)$ are fixed points under this involution (there are points not in $I(d)$ that are also fixed).

5.2.2 The involution $\#$ on $\mathfrak{N}$ and $\mathfrak{R}$

For $\alpha = (r, c)$ in $\mathfrak{N}$, or more generally in $\mathfrak{R}$, define $\alpha^{\#} = (c^*, r^*)$. The involution $\alpha \mapsto \alpha^{\#}$ is just the reflection with respect to the diagonal $\mathfrak{d}$. For a subset or even multiset $\mathfrak{S}$ of $\mathfrak{N}$ (or $\mathfrak{R}$), the symbol $\mathfrak{S}^{\#}$ has the obvious meaning. We call $\mathfrak{S}$ *symmetric* if $\mathfrak{S} = \mathfrak{S}^{\#}$.

Proposition 5.2.1 *An element $w \geq v$ of $I(d, 2d)$ belongs to $I(d)$ if and only if the distinguished subset $\mathfrak{S}_w$ of $\mathfrak{N}$ corresponding to it as described in §5.1.2 is symmetric and has evenly many diagonal elements.*

Proof: That the symmetry of $\mathfrak{S}_w$ is equivalent to the condition that $w = w^{\#}$ is proved in [4, Proposition 5.7]. Now suppose that $\mathfrak{S}_w$ is symmetric. We claim that for an element (r, c) of $\mathfrak{S}_w$ that is not on the diagonal, either both r and c are bigger than d or both are less than $d+1$. It is enough to prove the claim, for w is obtained from v by removing the column indices and adding the row indices of elements of $\mathfrak{S}_w$, and it would follow that the number of entries in w that are bigger than d equals the number of such entries in v plus the number of diagonal elements in $\mathfrak{S}_w$.

We now prove the claim. Since $\mathfrak{S}_w$ is symmetric, it follows that (c^*, r^*) also belongs to $\mathfrak{S}_w$. Since $\mathfrak{S}_w$ is distinguished, it follows that in case $r < c^*$ (that is, if (r, c) lies above the diagonal), we have $r < r^*$, and so $c < r < r^*$; and in case $r > c^*$, we have $c^* < c$, and so $c^* < c < r$. Thus the claim is proved. □

5.3 The subset $\mathfrak{S}_C$ attached to a v-chain C

5.3.1 Vertical and horizontal projections of an element of $\mathfrak{O}\mathfrak{N}$

For $\alpha = (r, c)$ in $\mathfrak{O}\mathfrak{N}$ (or more generally in $\mathfrak{O}\mathfrak{R}$), the elements $p_v(\alpha) := (c^*, c)$ and $p_h(\alpha) := (r, r^*)$ of the diagonal $\mathfrak{d}$ are called respectively the *vertical and horizontal projections of* α. In terms of pictures, the vertical projection is the element of the diagonal due South of α; the horizontal projection is the element of the diagonal due East of α. The vertical line joining α to its vertical projection $p_v(\alpha)$ and the horizontal line joining α to its horizontal projection $p_h(\alpha)$ are called the *legs* of α.

5.3.2 The "connection" relation on elements of a v-chain

Let $C : \alpha_1 = (r_1, c_1) > \alpha_2 = (r_2, c_2) > \cdots$ be a v-chain in $\mathfrak{O}\mathfrak{N}$. Two consecutive elements α_j and α_{j+1} of C are said to be *connected* if the following conditions are both satisfied:

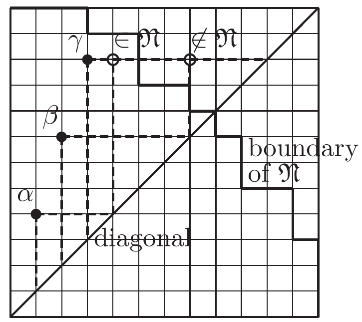
- their legs are "intertwined"; equivalently and more precisely, this means that $r_j^* \geq c_{j+1}$, or, what amounts to the same, $r_j \leq c_{j+1}^*$.

- the point (r_{j+1}, r_j^*) belongs to $\mathfrak{N}$; this just means that $r_{j+1} > r_j^*$.

Consider the coarsest equivalence relation on the elements of C generated by the above relation. The equivalence classes of C with respect to this equivalence relation are called the *connected components* of the v-chain C.

This definition has its quirks:

The v-chain $C : \alpha > \beta > \gamma$ in the picture has $\{\alpha, \beta\}$ and $\{\gamma\}$ as its connected components; but the "sub" v-chain $\alpha > \gamma$ of C is connected (as a v-chain in its own right).



5.3.3 The definition of $\mathfrak{S}_C$

We will define $\mathfrak{S}_C$ as a multiset of $\mathfrak{N}$. It is easy to see and in any case stated explicitly as part of Corollary 5.3.5 that it is multiplicity free and so is actually a subset of $\mathfrak{N}$.

First suppose that $C : \alpha_1 = (r_1, c_1) > \cdots > \alpha_\ell = (r_\ell, c_\ell)$ is a connected v-chain in $\mathfrak{O}\mathfrak{N}$. Observe that, if there is at all an integer j, $1 \leq j \leq \ell$, such that the horizontal projection $p_h(\alpha_j)$ does not belong to $\mathfrak{N}$, then $j = \ell$. Define

$$\mathfrak{S}_C := \begin{cases} \{p_v(\alpha_1), \ldots, p_v(\alpha_\ell)\} & \text{if } \ell \text{ is even} \\ \{p_v(\alpha_1), \ldots, p_v(\alpha_\ell)\} \cup \{p_h(\alpha_\ell)\} & \text{if } \ell \text{ is odd and } p_h(\alpha_\ell) \in \mathfrak{N} \\ \{p_v(\alpha_1), \ldots, p_v(\alpha_{\ell-1})\} \cup \{\alpha_\ell, \alpha_\ell^\#\} & \text{if } \ell \text{ is odd and } p_h(\alpha_\ell) \notin \mathfrak{N} \end{cases}$$

For a v-chain C that is not necessarily connected, let $C = C_1 \cup C_2 \cup \cdots$ be the partition of C into its connected components, and set

$$\mathfrak{S}_C := \mathfrak{S}_{C_1} \cup \mathfrak{S}_{C_2} \cup \cdots$$

5.3.4 The type of an element α of a v-chain C, and the set $\mathfrak{S}_{C,\alpha}$

We introduce some terminology and notation. Their usefulness may not be immediately apparent.

Suppose that $C : \alpha_1 > \cdots > \alpha_\ell$ is a connected v-chain. We define the *type in* C of an element α_j, $1 \leq j \leq \ell$, of C to be V, H, or S, accordingly as:

V: $j \neq \ell$, or $j = \ell$ and ℓ is even.

H: $j = \ell$, ℓ is odd, and $p_h(\alpha_\ell) \in \mathfrak{N}$.

S: $j = \ell$, ℓ is odd, and $p_h(\alpha_\ell) \notin \mathfrak{N}$.

The *type* of an element in a v-chain that is not necessarily connected is defined to be its type in its connected component.

The set $\mathfrak{S}_{C,\alpha}$ of elements of $\mathfrak{N}$ *generated* by an element α of C is defined to be:

$$\mathfrak{S}_{C,\alpha} := \begin{cases} \{p_v(\alpha)\} & \text{if } \alpha \text{ is of type V in } C; \\ \{p_v(\alpha), p_h(\alpha)\} & \text{if } \alpha \text{ is of type H in } C; \\ \{\alpha, \alpha^{\#}\} & \text{if } \alpha \text{ is of type S in } C; \end{cases}$$

Observe that, for a v-chain C, the monomial $\mathfrak{S}_C$ defined in §5.3.3 is the union, over all elements α of C, of $\mathfrak{S}_{C,\alpha}$.

For an element α of a v-chain C, we define $q_{C,\alpha}$ to be $p_v(\alpha)$ if α is of type V or H and to be α if it is of type S.

If the horizontal projection of an element in a v-chain does not belong to $\mathfrak{N}$, then clearly the same is true for every succeeding element. The first such element of a v-chain is called the *critical* element.

Proposition 5.3.1

1. *The cardinality is odd of a connected component that has an element of type H or S. Conversely, if the cardinality of a component is odd, then it has an element of type H or S.*

2. *An element of type H or S can only be the last element in its connected component.*

3. *The critical element has type either V or S. No element before it can be of type S and every element after it is of type S. In particular, any element that succeeds an element of type S is of type S.*

Proof: Clear from definitions. □

Proposition 5.3.2 *Let $\alpha > \gamma$ be elements of a v-chain C (we are not assuming that they are consecutive).*

1. *If $\alpha > \gamma$ is connected as a v-chain in its own right, then α is connected to its next member in C; that is, α cannot be the last element in its connected component in C.*

2. *If $\alpha > \gamma$ is not connected as a v-chain in its own right and the legs of α and γ intertwine, then the connected component of γ in C is the singleton $\{\gamma\}$, and γ has type S in C.*

Proof: Clear from definitions. □

Proposition 5.3.3 *Let $E : \alpha > \ldots > \zeta$ be a v-chain, D and D' two v-chains with tail α, and C, C' the concatenations of D, D' respectively with E. Then*

1. *The last element in the connected component containing α is the same in C and C' (and this is the same as in E).*

Let λ denote this element.

2. *The only element among α, ..., ζ that possibly has different types in C and C' is λ.*

Proof: (1): Whether or not two successive elements in a v-chain are connected is independent of other elements in the v-chain.

(2): The type of an element in a v-chain is V unless it is the last element in its connected component. And the type of the last element in a component depends on the cardinality of the component. The components of E not containing α are still components in C and C'. In contrast, the component containing α could possibly be larger in C (respectively C') and hence its cardinality could be different. $\square$

For an element $\alpha = (r, c)$ of $\mathfrak{N}$, we define $\alpha(\text{up})$ to be α itself if α is either on or above the diagonal $\mathfrak{d}$ (more precisely, if $r \leq c^*$), and to be its "reflection" in the diagonal (more precisely, (c^*, r^*)) if α is below the diagonal (more precisely, if $r > c^*$). For a monomial $\mathfrak{S}$ of $\mathfrak{N}$, $\mathfrak{S}(\text{up})$ is defined to be the intersection of $\mathfrak{S}$ (as a multiset) with the subset $\mathfrak{O}\mathfrak{N} \cup \mathfrak{d}$ of $\mathfrak{N}$. The notations $\alpha(\text{down})$ and $\mathfrak{S}(\text{down})$ have similar meanings.

> Caution: It is not true that $\mathfrak{S}(\text{up}) = \{\alpha(\text{up}) | \alpha \in \mathfrak{S}\}$ (in the obvious sense one would make of the right hand side). In particular, for a singleton monomial $\{\alpha\}$, it is not always true that $\{\alpha\}(\text{up}) = \{\alpha(\text{up})\}$.

Proposition 5.3.4 *Let α and β be elements of a v-chain C. Let us use α' and β' respectively to denote elements of $\mathfrak{S}_{C,\alpha}(\text{up})$ and $\mathfrak{S}_{C,\beta}(\text{up})$.*

1. *If $\alpha > \beta$ (these elements are not necessarily consecutive in C), then, given β', there exists α' such that $\alpha' > \beta'$. In fact, this is true for every choice of α' except when*

   (*) α is of type H, and $p_h(\alpha) \not> \beta'$ for some $\beta' \in \mathfrak{S}_{C,\beta}$.

   In particular, $q_{C,\alpha} > \beta'$ and $q_{C,\alpha} > q_{C,\beta}$.

2. *Conversely, suppose that $\alpha' > \beta'$ for some choice of α' and β'. Then $\alpha \geq \beta$; if equality occurs, then α is of type H, $\alpha' = p_v(\alpha)$ and $\beta' = p_h(\alpha)$. In particular, if $\alpha' > q_{C,\beta}$ (or more specially $q_{C,\alpha} > q_{C,\beta}$), then $\alpha > \beta$.*

3. *If (*) holds for $\alpha > \beta$ in C, then*

   (a) *the critical element of C is the one just after α; in particular, α is uniquely determined.*

   (b) *all elements of C succeeding α are of type S; in particular, β is of type S and $\beta' = \beta$.*

   (c) *(*) holds for γ in place of β for every γ in C that succeeds α.*

Proof: (1) If α is of type V or H, we need only take $\alpha' = p_v(\alpha)$, for $p_v(\alpha) > p_v(\beta)$, $p_v(\alpha) > p_h(\beta)$, and $p_v(\alpha) > \beta$. Now suppose that α is of type S. Then β too is of type S (Proposition 5.3.1 (3)), so β' can only be β, and the first part of (1) is proved.

It follows from the above that if $\alpha' = p_v(\alpha)$ or if α has type S, then $\alpha' > \beta'$ independent of the choice of α'. So if $\alpha' \not> \beta'$, then (*) holds and $\alpha' = p_h(\alpha)$.

(3) Let λ be the immediate successor of α in C. Then α is not connected to λ (Proposition 5.3.1 (2)). Since $p_h(\alpha) \not> \beta'$, it follows that α and β have intertwining legs. Therefore so do α and λ. By Proposition 5.3.2 (2), λ has type S in C.

Since α has type H and λ type S, it follows immediately from the definition of the critical element that λ is the critical element. This proves (a). Assertion (b) now follows from Proposition 5.3.1 (3). For (c), write $p_h(\alpha) = (a, a^*)$, $\lambda = (R, C)$, and $\gamma = (r, c)$. Then $R < a^*$, for α and λ have intertwining legs but are not connected. So $c < r \leq R < a^*$. This means $p_h(\alpha) \not> \gamma$. And γ being of type S (by (b)), we can take $\gamma' = \gamma$.

(2) Suppose that $\alpha \not\geq \beta$. Then $\beta > \alpha$. By the second part of (1) above, β is of type H and $\beta' = p_h(\beta)$; by item (b) of (3), α is of type S, so $\alpha' = \alpha$. This leads to the contradiction $\beta > \alpha > p_h(\beta)$. $\square$

Corollary 5.3.5 *The multiset $\mathfrak{S}_C$ attached to a v-chain C is a distinguished subset of $\mathfrak{N}$ in the sense of 5.1.1.*

Proof: If α in C is of type V or S, then $\mathfrak{S}_{C,\alpha}$ is a singleton; if it is of type H, then $\mathfrak{S}_{C,\alpha} = \{p_v(\alpha), p_h(\beta)\}$. So there can be no violation of conditions A and B of §5.1.1 by elements of $\mathfrak{S}_{C,\alpha}$.

Suppose $\alpha > \beta$. By Proposition 5.3.4 (1), we have $\alpha' > \beta'$ for any choice of $\alpha' \in \mathfrak{S}_{C,\alpha}$ and $\beta' \in \mathfrak{S}_{C,\beta}$ except when the condition (*) holds. By (3) of the same proposition, if (*) holds, then $\beta' = \beta$, and writing $\beta = (r, c)$, $p_h(\alpha) = (a, a^*)$, we have $r < a$ (since $\alpha > \beta$) and $c < r < a^*$ (see proof of item 3(c) of the proposition). Thus there can be no violation of conditions A and B of §5.1.1. $\square$

Corollary 5.3.6 *Let $\mathfrak{S}$ be a v-chain in $\mathfrak{O}\mathfrak{N}$ and w an element of $I(d)$. If w $\mathfrak{O}$-dominates $\mathfrak{S}$, then w dominates in the sense of* [7] *the monomial $\mathfrak{S} \cup \mathfrak{S}^\#$ of $\mathfrak{N}$.*

Proof: By [4, Proposition 5.15], it is enough to show that w dominates $\mathfrak{S}$. Let $C : \alpha_1 > \ldots > \alpha_t$ be a v-chain in $\mathfrak{S}$. Writing $\alpha_j = (r_j, c_j)$ and $q_{C,\alpha_j} = (R_j, C_j)$ we have $r_j \leq R_j$ and $C_j \leq c_j$. By Proposition 5.3.4 (1), we have $q_{C,\alpha_1} > \ldots > q_{C,\alpha_t}$. Since w $\mathfrak{O}$-dominates $\mathfrak{S}$, it in particular dominates $q_{C,\alpha_1} > \ldots > q_{C,\alpha_t}$ and so also C. $\square$

6 $\mathfrak{O}$-depth

The concept of $\mathfrak{O}$-depth defined in §6.1 below plays a key role in this paper. As the name suggests, it is the orthogonal analogue of the concept of depth of [7]. In §6.2 below, it is observed that the $\mathfrak{O}$-depth is no smaller than depth in the sense of [7]. In §6.3, some observations about the relation between $\mathfrak{O}$-depths and types of elements in v-chains are recorded.

6.1 Definition of $\mathfrak{O}$-depth

The *$\mathfrak{O}$-depth of an element α in a v-chain C in $\mathfrak{O}\mathfrak{N}$* is the depth in $\mathfrak{S}_C$ in the sense of [7] of $q_{C,\alpha}$: in other words, it is the depth in $\mathfrak{S}_C$ of $p_v(\alpha)$ in case α is of type V or H, and of α (equivalently of $\alpha^\#$) in case α is of type S. It is denoted $\mathfrak{O}$-depth$_C(\alpha)$. The *$\mathfrak{O}$-depth of an element α in a monomial $\mathfrak{S}$ of $\mathfrak{O}\mathfrak{N}$* is the maximum, over all v-chains C in $\mathfrak{S}$ containing α, of the $\mathfrak{O}$-depth of α in C. It is denoted $\mathfrak{O}$-depth$_\mathfrak{S}(\alpha)$. Finally, the *$\mathfrak{O}$-depth of a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$* is the maximum of the $\mathfrak{O}$-depths in $\mathfrak{S}$ of all the elements of $\mathfrak{S}$.

There is a conflict in the above definitions: Is the $\mathfrak{O}$-depth of an element of a v-chain C the same as its depth as an element of the monomial C? In other words, could the $\mathfrak{O}$-depth of an element in a v-chain be exceeded by its $\mathfrak{O}$-depth in a sub-chain? The conflict is resolved by the first item of the following proposition.

Proposition 6.1.1 *1. For v-chains $C \subseteq D$, the $\mathfrak{O}$-depth in C of an element of C is no more than its $\mathfrak{O}$-depth in D.*

2. If a v-chain C is an initial segment of a v-chain D, then the $\mathfrak{O}$-depths in C and D of an element of C are the same.

PROOF: (1): By an induction on the difference in the cardinalities of D and C, we may assume that D has one more element than C. Call this extra element δ. Suppose that δ lies between successive elements α and β of C (the modifications needed to cover the extreme cases when it goes at the beginning or the end are being left to the reader).

The only elements of C that could possibly undergo changes of type on addition of δ are α and the last element in the connected component of β, which let us call β'. If there are no type changes, then $\mathfrak{S}_C \subseteq \mathfrak{S}_D$ and the assertion is immediate. The only type change that α can undergo is from H to V. The type changes that β' can undergo are: H to V; V to H; S to V; V to S. An easy enumeration of cases shows that only one of α and β' can undergo a type change.

We need not worry about changes from V to H for in this case $\mathfrak{S}_C \subseteq \mathfrak{S}_D$.

First let us suppose that α undergoes a change of type (from H to V). Then δ is connected to α. It follows from Proposition 5.3.1 (1) that δ has type V in D: the connected component of α in C has odd number of elements, so if δ happens to be the last element in its connected component in D, the number of elements in that component will be even. Replacing an occurrence of $p_h(\alpha)$ in a v-chain

of $\mathfrak{S}_C$ by $p_v(\delta)$ would result in a v-chain in $\mathfrak{S}_D$ (by Proposition 5.3.4 (1)), and this case is settled.

Now suppose that β' undergoes a type change. Then δ is connected to β and δ is of type V in D (Proposition 5.3.1 (2)). Replacing by $p_v(\delta)$ any occurrence in a v-chain in $\mathfrak{S}_C$ of $p_v(\beta')$, $p_h(\beta')$, β' accordingly as the type of β' in C is V, H, or S, (not necessarily in the same place but at an appropriate place) would result in a v-chain in $\mathfrak{S}_D$ (by Proposition 5.3.4 (1)), and we see that the $\mathfrak{O}$-depth cannot decrease.

(2): It follows from Proposition 5.3.4 (2) that, for an element α of C, contributions to $\mathfrak{S}_D$ from elements beyond α (in particular from those not in C) do not affect the depth in $\mathfrak{S}_D$ of $q_{D,\alpha}$. Looking for the possibility of differences in types in C and D of elements of C, we see that the only element of C that has possibly a different type in D is its last element. And this too can change type only from H to V.

The above two observations imply that the calculations of $\mathfrak{O}$-depths in C and D of an element α of C are no different: we would be considering the depth in $\mathfrak{S}_C$ and $\mathfrak{S}_D$ respectively of the same element (either $p_v(\alpha)$ or α), and the differences in $\mathfrak{S}_D$ and $\mathfrak{S}_C$ have no effect on this consideration. $\square$

Corollary 6.1.2 *If $C \subseteq D$ are v-chains in $\mathfrak{O}\mathfrak{N}$, then $w_C \leq w_D$ (although it is not always true that $\mathfrak{S}_C \subseteq \mathfrak{S}_D$).*

Proof: By [7, Lemma 5.5], it is enough to show that every v-chain in $\mathfrak{S}_C$ is dominated by w_D. Let $\beta_1 = (r_1, c_1) > \cdots > \beta_t = (r_t, c_t)$ be an arbitrary v-chain in $\mathfrak{S}_C$. To show that it is dominated by w_D, it is enough, by [7, Lemma 4.5], to show the existence of a v-chain $(R_1, C_1) > \cdots > (R_t, C_t)$ in $\mathfrak{S}_D$ with $r_j \leq R_j$ and $C_j \leq c_j$ for $1 \leq j \leq t$. Such a v-chain exists by the proof of (1) of Proposition 6.1.1. $\square$

Corollary 6.1.3

1. *Let $\mathfrak{S}$ be a monomial in $\mathfrak{O}\mathfrak{N}$ and $\alpha \in \mathfrak{S}$. Then there exists a v-chain C in $\mathfrak{S}$ with tail α such that $\mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha) = \mathfrak{O}\text{-depth}_C(\alpha)$.*
2. *For elements $\alpha > \gamma$ in a v-chain C (these need not be consecutive), we have $\mathfrak{O}\text{-depth}_C(\alpha) < \mathfrak{O}\text{-depth}_C(\gamma)$.*
3. *For elements $\alpha > \gamma$ of a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$, we have $\mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha) < \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\gamma)$.*
4. *No two elements of the same $\mathfrak{O}$-depth in a monomial in $\mathfrak{O}\mathfrak{N}$ are comparable.*

Proof: (1) This follows from (2) of the Proposition above and the definition of $\mathfrak{O}$-depth.

(2) This follows from Proposition 5.3.4 (1) and the definition of $\mathfrak{O}$-depth.

(3) By (1), there exists a v-chain C with tail α such that $\mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha) = \mathfrak{O}\text{-depth}_C(\alpha)$. Concatenate C with $\alpha > \gamma$ and let D denote the resulting v-chain.

By (2) of the Proposition above, $\mathfrak{O}\text{-depth}_C(\alpha) = \mathfrak{O}\text{-depth}_D(\alpha)$. By (2) above, $\mathfrak{O}\text{-depth}_D(\alpha) < \mathfrak{O}\text{-depth}_D(\gamma)$. And finally, $\mathfrak{O}\text{-depth}_D(\gamma) \leq \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\gamma)$ by the definition of $\mathfrak{O}\text{-depth}_{\mathfrak{S}}(\gamma)$.

(4) Immediate from (3). $\square$

Corollary 6.1.4 *Let $\beta > \gamma$ be elements of a v-chain C of elements of $\mathfrak{O}\mathfrak{N}$. Let E be a v-chain in $\mathfrak{S}_C$ with tail $q_{C,\gamma}$ and length* $\mathfrak{O}\text{-depth}_C(\gamma)$. *Then $q_{C,\beta}$ occurs in E.*

Proof: It is enough to show that for $\alpha' \neq q_{C,\beta}$ in E, either $\alpha' > q_{C,\beta}$ or $q_{C,\beta} > \alpha'$. Let α be in C such that $q_{C,\beta} \neq \alpha' \in \mathfrak{S}_{C,\alpha}$. If $\beta \geq \alpha$, then $q_{C,\beta} > \alpha'$ by Proposition 5.3.4 (1). If $\alpha > \beta$ and $\alpha' \not> q_{C,\beta}$, then, by (1) and (3) of the same proposition, $\alpha' \not> q_{C,\gamma}$, a contradiction. $\square$

6.2 $\mathfrak{O}$-depth and depth

Lemma 6.2.1 *The $\mathfrak{O}$-depth of an element α in a monomial $\mathfrak{S}$ of $\mathfrak{O}\mathfrak{N}$ is no less than its depth (in the sense of* [7]*) in $\mathfrak{S} \cup \mathfrak{S}^{\#}$.*

Proof: Let $C : \alpha_1 > \ldots > \alpha_t$ be a v-chain in $\mathfrak{S} \cup \mathfrak{S}^{\#}$ with tail $\alpha_t = \alpha$, where t is the depth of α in $\mathfrak{S} \cup \mathfrak{S}^{\#}$. We then have $\alpha_1(\text{up}) > \ldots > \alpha_t(\text{up})$, so we may assume C to be in $\mathfrak{S}$. By Proposition 5.3.4 (1), $q_{C,\alpha_1} > \ldots > q_{C,\alpha_t}$ in $\mathfrak{S}_C$. So $\text{depth}_{\mathfrak{S}\cup\mathfrak{S}^{\#}}(\alpha) = t \leq \text{depth}_{\mathfrak{S}_C}(q_{C,\alpha_t}) \leq \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha)$. $\square$

6.3 $\mathfrak{O}$-depth and type

We begin by defining some useful terminology. Let (r, c) and (R, C) be two elements of $\mathfrak{R}$. To say that (R, C) *dominates* (r, c) means that $r \leq R$ and $C \leq c$ (in terms of pictures, (r, c) lies (not necessarily strictly) to the Northeast of (R, C)). To say that they are *comparable* means that either $(R, C) > (r, c)$ or $(r, c) > (R, C)$. While this is admittedly strange, there will arise no occasion for confusion.

For an integer i, we let $i(\text{odd})$ be the largest odd integer not bigger than i and $i(\text{even})$ the smallest even integer not smaller than i.

Lemma 6.3.1 1. *For consecutive elements $\alpha > \beta$ of a v-chain C,*

$$\mathfrak{O}\text{-depth}_C(\beta) = \begin{cases} \mathfrak{O}\text{-depth}_C(\alpha) + 2 & \textit{if and only if } \alpha \textit{ is of} \\ & \textit{type } H \textit{ and } p_h(\alpha) > \beta \\ \mathfrak{O}\text{-depth}_C(\alpha) + 1 & \textit{otherwise} \end{cases}$$

2. *For an element of a v-chain C such that either its horizontal projection belongs to $\mathfrak{N}$ or it is connected to its predecessor, the parity of its $\mathfrak{O}$-depth in C is the same as that of its ordinality in its connected component in C.*

3. *The $\mathfrak{O}$-depth in a v-chain of an element of type H is odd.*

4. *If in a v-chain an element of type V is the last in its connected component, then its $\mathfrak{O}$-depth is even.*

5. *If in a v-chain C there is an element of $\mathfrak{O}$-depth d, then*

   (a) *for every odd integer d' not exceeding d, there is in C an element of $\mathfrak{O}$-depth d'.*

   (b) *if, for an even integer d' not exceeding d, there is no element in C of $\mathfrak{O}$-depth d', then the element α in C of $\mathfrak{O}$-depth $d'-1$ is of type H, and $p_h(\alpha) > \beta$, where β denotes the immediate successor of α in C.*

6. *Let C be a v-chain and α an element of type H in C. Then the depth in $\mathfrak{S}_C$ of $p_h(\alpha)$ equals $\mathfrak{O}\text{-depth}_C(\alpha)+1$. In particular, this depth is even.*

PROOF: (1): From items 1 and 3(a) of Proposition 5.3.4, it follows that, for γ in C with $\gamma > \alpha$, if $\gamma' \not> q_{C,\alpha}$ for some γ' in $\mathfrak{S}_{C,\gamma}$, then $\gamma' \not> q_{C,\beta}$. Thus $\mathfrak{O}\text{-depth}_C(\beta)$ exceeds $\mathfrak{O}\text{-depth}_C(\alpha)$ by the number of elements in $\mathfrak{S}_{C,\alpha}$ that dominate $q_{C,\beta}$. This number is 1 if α is of type V, or of type S, or of type H and $p_h(\alpha) \not> \beta$; it is 2 if α is of type H and $p_h(\alpha) > \beta$ (note that $p_h(\alpha) > \beta$ if and only if $p_h(\alpha) > q_{C,\beta}$).

(2): Let λ be such an element. Everything preceding λ in C is of type H or V (Proposition 5.3.1 (3)). Let λ belong to the k^{th} connected component, and $n_1, \ldots, n_k$ be respectively the cardinalities of the first, …, k^{th} connected components. By (1) above and item 3(b) of Proposition 5.3.4, $\mathfrak{O}\text{-depth}_C(\lambda)$ is $n_1(\text{even}) + \cdots + n_{k-1}(\text{even})$ plus the ordinality of λ in the k^{th} connected component.

(3) and (4): These are special cases of 2.

(5): This follows easily from (1) and (3).

(6): It follows from Proposition 5.3.4 (2) that there is no element γ in $\mathfrak{S}_C$ that lies between $p_v(\alpha)$ and $p_h(\alpha)$ (meaning $p_v(\alpha) > \gamma > p_h(\alpha)$), so the assertion holds. $\square$

Corollary 6.3.2 *For a v-chain C in $\mathfrak{O}\mathfrak{N}$, if the $\mathfrak{O}$-depths of elements in C are bounded by k, then the depths of elements in $\mathfrak{S}_C$ are bounded by $k(\text{even})$.*

PROOF: The depth of $q_{C,\alpha}$ in $\mathfrak{S}_C$ for any α in C is at most k by hypothesis. An element of $\mathfrak{S}_C$ that is not $q_{C,\alpha}$ for any α in C can only be of the form $p_h(\alpha)$ for some α. By Proposition 5.3.4, $\text{depth}_{\mathfrak{S}_C} p_v(\alpha) = \text{depth}_{\mathfrak{S}_C} p_h(\alpha) - 1$, which implies $\text{depth}_{\mathfrak{S}_C} p_h(\alpha) \le k+1$. If, moreover, k is even, then by (3) of Lemma 6.3.1 $\text{depth}_{\mathfrak{S}_C} p_h(\alpha) = \text{depth}_{\mathfrak{S}_C} p_v(\alpha) + 1 \le (k-1)+1 = k$. $\square$

Proposition 6.3.3 *Given a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$ and an element α in it, there exists a v-chain C in $\mathfrak{S}$ with tail α such that $\mathfrak{O}\text{-depth}_C(\beta) = \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\beta)$ for every β in C.*

PROOF: Proceed by induction on $d := \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha)$. Choose a v-chain D in $\mathfrak{S}$ with tail α such that $\mathfrak{O}\text{-depth}_D(\alpha) = \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha)$ (such a v-chain exists by Corollary 6.1.3 (1)). Let α' be the element in D just before α. It follows from item (3) of Corollary 6.1.3 and item (1) of Lemma 6.3.1 that $\mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha')$ (as also $\mathfrak{O}\text{-depth}_D(\alpha')$) is either $d-1$ or $d-2$. By induction, there exists a v-chain C' with tail α' that has the desired property. Let C be the concatenation of C' with $\alpha' > \alpha$.

We claim that C has the desired property. The only thing to be proved is that $\mathfrak{O}\text{-depth}_C(\alpha) = d$. By item (1) of Lemma 6.3.1, we have $\mathfrak{O}\text{-depth}_C(\alpha) \geq \mathfrak{O}\text{-depth}_{C'}(\alpha') + 1$. In particular, the claim is proved in case $\mathfrak{O}\text{-depth}_{C'}(\alpha')$ is $d-1$, so let us assume that $\mathfrak{O}\text{-depth}_{C'}(\alpha')$ is $d-2$. It now follows from the same item that α' has type H in D and $p_h(\alpha') > \alpha$; it further follows that it is enough to show that α' has type H in C.

Since α' has type H in D, it follows (from item (2) of Proposition 5.3.1) that $\alpha' > \alpha$ is not connected and (from item (3) of Lemma 6.3.1) that $d-2$ is odd. Now, by item (4) of Lemma 6.3.1, the type in C' of α' cannot be V, so it is H, and the claim is proved. □

Corollary 6.3.4 *Let $\mathfrak{S}$ be a monomial in $\mathfrak{OM}$, β an element of $\mathfrak{S}$, and i an integer such that $i < \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\beta)$. Then*

(a) If i is odd, there exists an element α in $\mathfrak{S}$ of $\mathfrak{O}$-depth i such that $\alpha > \beta$.

(b) If i is even and there is no element α in $\mathfrak{S}$ of $\mathfrak{O}$-depth i such that $\alpha > \beta$, then there is element α in $\mathfrak{S}$ of $\mathfrak{O}$-depth $i-1$ such that $p_h(\alpha) > \beta$.

PROOF: Choose a v-chain C in $\mathfrak{S}$ having tail β and the good property of Proposition 6.3.3. Apply Lemma 6.3.1 (5). □

Corollary 6.3.5 *Let C be a v-chain in $\mathfrak{OM}$ with tail α such that $\mathfrak{O}\text{-depth}_C(\alpha)$ is odd. Let A be a v-chain in $\mathfrak{OM}$ with head α, and D the concatenation of C with A. Let C' denote the v-chain $C \setminus \{\alpha\}$. Then*

1. *The type of an element of A is the same in both A and D. In particular, $\mathfrak{S}_A \subseteq \mathfrak{S}_D$ and $q_{A,\beta} = q_{D,\beta}$ for β in A.*

2. *The type of an element of C' is the same in both C' and D. In particular, $\mathfrak{S}_{C'} \subseteq \mathfrak{S}_D$.*

3. *$\mathfrak{S}_D = \mathfrak{S}_{C'} \cup \mathfrak{S}_A$ (disjoint union); letting $j_0 := \mathfrak{O}\text{-depth}_C(\alpha)$ we have $(\mathfrak{S}_D)^{j_0} = \mathfrak{S}_A$ and $(\mathfrak{S}_D)_1 \cup \cdots \cup (\mathfrak{S}_D)_{j_0-1} = \mathfrak{S}_{C'}$. (For a monomial $\mathfrak{S}$, the subset of elements of depth at least i is denoted $\mathfrak{S}^i$, and the subset of elements of depth exactly i is denoted $\mathfrak{S}_i$.)*

PROOF: (1) Generally (meaning without the assumption that $\mathfrak{O}\text{-depth}_C(\alpha)$ is odd), the only element of A that could possibly have a different type in D is the last one in the first connected component of A; whether or not it changes

type depends exactly upon whether or not the parity of the cardinality of its connected component in D is different from that in A. Under our hypothesis, this parity does not change, for, by (4) of Lemma 6.3.1, the type of α in C is H or S, and so the cardinality of the connected component of α in C is odd.

(2) Generally (meaning without the assumption that $\mathfrak{O}$-depth$_C(\alpha)$ is odd), the only element of C' that could possibly have a different type in D is the last one of C'; it changes type if and only if it is connected to α and the cardinality of its connected component in C' is odd. Under our hypothesis, this cardinality is even, for the same reason as in (1).

(3) That $\mathfrak{S}_D = \mathfrak{S}_{C'} \cup \mathfrak{S}_A$ (disjoint union) is an immediate consequence of (1) and (2). By Lemma 6.3.1 (1), $q_{A,\alpha} = q_{D,\alpha}$ dominates every element of $\mathfrak{S}_A$, so $\mathfrak{S}_A \subseteq (\mathfrak{S}_D)^{j_0}$ (depth$_{\mathfrak{S}_D} q_{D,\alpha} = \mathfrak{O}$-depth$_D(\alpha) = \mathfrak{O}$-depth$_C(\alpha) = j_0$). It is enough to prove the following claim: every element of $\mathfrak{S}_{C'}$ has depth less than j_0 in $\mathfrak{S}_D$. Let γ' be an element of $\mathfrak{S}_{C'}$. If $\gamma' > q_{D,\alpha}$ then the claim is clear. If not, then, by Proposition 5.3.4 (1), $\gamma' = p_h(\gamma)$. By Lemma 6.3.1 (3), $\mathfrak{O}$-depth$_D(\gamma)$ is odd. Since the claim is already true for $q_{D,\gamma} = p_v(\gamma)$, we have $\mathfrak{O}$-depth$_D(\gamma) = $ depth$_{\mathfrak{S}_D} p_v(\gamma) \leq j_0 - 2$. By (6) of the same lemma, depth$_{\mathfrak{S}_D}\gamma' = \mathfrak{O}$-depth$_D(\gamma)+1$, so depth$_D\gamma' \leq j_0 - 1$, and the claim is proved.□

Proposition 6.3.6 *Let $\mathfrak{S}$ be a monomial in $\mathfrak{O}\mathfrak{N}$ and j an odd integer. For β in $\mathfrak{S}^{j,j+1}(:= \{\alpha \in \mathfrak{S} \,|\, \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha) \geq j\})$, we have*

$$\mathfrak{O}\text{-depth}_{\mathfrak{S}^{j,j+1}}(\beta) = \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\beta) - j + 1$$

Proof: Proceed by induction on j. For $j = 1$, the assertion reduces to a tautology. Suppose that the assertion has been proved upto j. By the induction hypothesis, we have $\mathfrak{S}^{j+2,j+3} = (\mathfrak{S}^{j,j+1})^{3,4}$, and we are reduced to proving the assertion for $j = 3$.

Let A be a v-chain in $\mathfrak{S}^{3,4}$ with tail β and $\mathfrak{O}$-depth$_A(\beta) = \mathfrak{O}$-depth$_{\mathfrak{S}^{3,4}}(\beta)$. Let α be the head of A. We may assume that $\mathfrak{O}$-depth$_{\mathfrak{S}}(\alpha) = 3$ for, if $\mathfrak{O}$-depth$_{\mathfrak{S}}(\alpha) > 3$, we can find, by Lemma 6.3.1 (5), α' of $\mathfrak{O}$-depth 3 in $\mathfrak{S}$ with $\alpha' > \alpha$, and extending A by α' will not decrease the $\mathfrak{O}$-depth in A of β (Proposition 6.1.1 (1)). Let E be a v-chain in $\mathfrak{S}_A$ with tail $q_{A,\beta}$ and length $\mathfrak{O}$-depth$_A(\beta)$. The head of E is then $q_{A,\alpha}$ (see Proposition 5.3.4 (1)).

Choose C in $\mathfrak{S}$ with tail α such that $\mathfrak{O}$-depth$_C(\alpha) = 3$. Let D be the concatenation of C with A. By Corollary 6.3.5, E is contained in $\mathfrak{S}_D$, $q_{D,\alpha} = q_{A,\alpha}$, and $q_{D,\beta} = q_{A,\beta}$. By Proposition 6.1.1 (2), the $\mathfrak{O}$-depth of α is the same in D as in C. Choose a v-chain F in $\mathfrak{S}_D$ with tail $q_{D,\alpha} = q_{A,\alpha}$. Concatenating F with E we get a v-chain in $\mathfrak{S}_D$ with tail $q_{D,\beta} = q_{A,\beta}$ of length $\mathfrak{O}$-depth$_{\mathfrak{S}^{3,4}}(\beta)+2$. This proves that $\mathfrak{O}$-depth$_{\mathfrak{S}}(\beta) \geq \mathfrak{O}$-depth$_{\mathfrak{S}^{3,4}}(\beta) + 2$.

To prove the reverse inequality, we need only turn the above proof on its head. Let D be a v-chain in $\mathfrak{S}$ with tail β such that $\mathfrak{O}$-depth$_{\mathfrak{S}}(\beta) = \mathfrak{O}$-depth$_D(\beta)$. Let G be a v-chain in $\mathfrak{S}_D$ with tail $q_{D,\beta}$ and length $\mathfrak{O}$-depth$_{\mathfrak{S}}(\beta)$. There exists an element α in D of $\mathfrak{O}$-depth 3 in D (by Lemma 6.3.1 (5)). Let C be the part of D upto and including α, and A the part $\alpha > \ldots > \beta$. By Proposition 6.1.1 (2), $\mathfrak{O}$-depth$_C(\alpha) = 3$ and, as above, Corollary 6.3.5 applies.

By Corollary 6.1.4, $q_{A,\alpha} = q_{D,\alpha}$ occurs in G. The part F of G upto and including $q_{A,\alpha}$ is of length at most 3, and the part $E : q_{D,\alpha} > \ldots > q_{D,\beta}$ belongs also to $\mathfrak{S}_A$ (Proposition 5.3.4 (2)). Thus the length of G is at most 2 more than the the length of E which is at most $\mathfrak{O}\text{-depth}_{\mathfrak{S}^{3,4}}(\beta)$. □

Corollary 6.3.7 *For odd integers i, j, we have $(\mathfrak{S}^{i,i+1})^{j,j+1} = \mathfrak{S}^{i+j-1,i+j}$.* □

Corollary 6.3.8 *Let $E : \alpha > \ldots > \zeta$ be a v-chain, D and D' two v-chains with tail α, and C, C' the concatenations of D, D' respectively with E. Then*

1. $\mathfrak{O}\text{-depth}_C(\zeta) - \mathfrak{O}\text{-depth}_C(\alpha) \leq \mathfrak{O}\text{-depth}_{C'}(\zeta) - \mathfrak{O}\text{-depth}_{C'}(\alpha) + 1$*;*
2. *equality holds if and only if the type of λ is H in C and V in C', and $p_h(\lambda) > \mu$, where λ is the last element in the connected component containing α of E and μ is the immediate successor in E of λ.*

Proof: These assertions follow from combining (2) of Proposition 5.3.3 with (1) of Lemma 6.3.1. □

Corollary 6.3.9 *Let ζ be an element of a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$. Let C be a v-chain in $\mathfrak{S}$ with tail ζ such that $\mathfrak{O}\text{-depth}_C(\zeta) = \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\zeta)$. Then*

1. $\mathfrak{O}\text{-depth}_C(\alpha) \geq \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha) - 1$ *for any α in C.*
2. *If $\mathfrak{O}\text{-depth}_C(\alpha) = \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha) - 1$ for some α in C, then*
   (a) *letting λ be the last element in the connected component containing α and μ the element next to λ, the type of λ in C is H and $p_h(\lambda) > \mu$.*
   (b) $\mathfrak{O}\text{-depth}_C(\gamma) = \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\gamma) - 1$ *for all γ in C between α and λ (both inclusive).*

Proof: (1) Let α be in C. Let E denote the part of C beyond (and including) α. Let D' be a v-chain in $\mathfrak{S}$ with tail α such that $\mathfrak{O}\text{-depth}_{D'}(\alpha) = \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha)$. Let C' be the concatenation of D' and E. Applying Proposition 6.3.8 (1), we have

$$\mathfrak{O}\text{-depth}_C(\alpha) \geq \mathfrak{O}\text{-depth}_C(\zeta) - \mathfrak{O}\text{-depth}_{C'}(\zeta) + \mathfrak{O}\text{-depth}_{C'}(\alpha) - 1.$$

But $\mathfrak{O}\text{-depth}_C(\zeta) - \mathfrak{O}\text{-depth}_{C'}(\zeta) = \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\zeta) - \mathfrak{O}\text{-depth}_{C'}(\zeta) \geq 0$, and, by the choice of D' and Proposition 6.1.1 (2), $\mathfrak{O}\text{-depth}_{C'}(\alpha) = \mathfrak{O}\text{-depth}_{D'}(\alpha) = \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha)$.

(2) Assertions (a) and (b) follow respectively from the "only if " and "if" parts of item (2) of Proposition 6.3.8. □

7 The map $\mathfrak{O}\pi$

The purpose of this section is to describe the map $\mathfrak{O}\pi$. The description is given in §7.1. It relies on certain claims which are proved in §§7.3, 7.4. Those proofs in turn refer to results from §9, but there is no circularity—to postpone the definition of $\mathfrak{O}\pi$ until all the results needed for it have been proved would hurt rather than help readability. The observations in §7.5 are required only in §10.

The symbol j will be reserved for an odd positive integer throughout this section.

7.1 Description of $\mathfrak{O}\pi$

The map $\mathfrak{O}\pi$ takes as input a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$ and produces as output a pair $(w, \mathfrak{S}')$, where w is an element of $I(d)$ such that $w \geq v$ and $\mathfrak{S}'$ is a "smaller" monomial, possibly empty, in $\mathfrak{O}\mathfrak{N}$. If the input $\mathfrak{S}$ is empty, no output is produced (by definition). So now suppose that $\mathfrak{S}$ is non-empty.

We first partition $\mathfrak{S}$ into subsets according to the $\mathfrak{O}$-depths of its elements. Let $\mathfrak{S}_k^{\mathrm{pr}}$ be the sub-monomial of $\mathfrak{S}$ consisting of those elements of $\mathfrak{S}$ that have $\mathfrak{O}$-depth k—the superscript "pr" is short for "preliminary". It follows from Corollary 6.1.3 (4) that there are no comparable elements in $\mathfrak{S}_k^{\mathrm{pr}}$ and so we can arrange the elements of $\mathfrak{S}_k^{\mathrm{pr}}$ in ascending order of both row and column indices. Let σ_k be the last element of $\mathfrak{S}_k^{\mathrm{pr}}$ in this arrangement.

Let now j be an odd integer. We set

$$\mathfrak{S}_{j,j+1}^{\mathrm{pr}} := \mathfrak{S}_j^{\mathrm{pr}} \cup \mathfrak{S}_{j+1}^{\mathrm{pr}}.$$

We say that $\mathfrak{S}$ *is truly orthogonal at* j if $p_h(\sigma_j)$ belongs to $\mathfrak{N}$ (that is, if $r > r^*$ where $\sigma_j = (r, c)$),

Let $\mathfrak{S}_{j,j+1}$ denote the monomial in $\mathfrak{N}$ defined by $\mathfrak{S}_{j,j+1} :=$

$$\begin{cases} \left(\mathfrak{S}_{j,j+1}^{\mathrm{pr}} \setminus \{\sigma_j\}\right) \cup \left(\mathfrak{S}_{j,j+1}^{\mathrm{pr}} \setminus \{\sigma_j\}\right)^{\#} \cup \{p_v(\sigma_j), p_h(\sigma_j)\} & \text{if } \mathfrak{S} \text{ is truly orthogonal at } j \\ \mathfrak{S}_{j,j+1}^{\mathrm{pr}} \cup \left(\mathfrak{S}_{j,j+1}^{\mathrm{pr}}\right)^{\#} & \text{otherwise} \end{cases}$$

Here $\mathfrak{S}_{j,j+1}^{\mathrm{pr}} \setminus \{\sigma_j\}$ and other terms on the right are to be understood as multisets. As proved in Corollary 7.3.4 (1) below, $\mathfrak{S}_{j,j+1}$ has depth at most 2. Let $\mathfrak{S}_j$ (respectively $\mathfrak{S}_{j+1}$ be the subset (as a multiset) of elements of depth 1 (respectively 2) of $\mathfrak{S}_{j,j+1}$.

Now, for every integer k, we apply the map of π of [7, §4] to $\mathfrak{S}_k$ to obtain a pair $(w(k), \mathfrak{S}_k')$, where $w(k)$ is an element of $I(d, 2d)$ and $\mathfrak{S}_k'$ is a monomial in $\mathfrak{N}$. Let $\mathfrak{S}_{w(k)}$ be the distinguished monomial in $\mathfrak{N}$ associated to $w(k)$—see §5.1.2.

Proposition 7.1.1 *1. $\mathfrak{S}_{w(k)}$ and $\mathfrak{S}_k'$ are symmetric. And therefore so are $\cup_k \mathfrak{S}_{w(k)}$ and $\cup_k \mathfrak{S}_k'$.*

2. $\cup_k \mathfrak{S}_{w(k)}$ is a distinguished subset of $\mathfrak{N}$ (in particular, the $\mathfrak{S}_{w(k)}$ are disjoint).

3. *For j an odd integer, either*
   - *both $\mathfrak{S}_{w(j)}$ and $\mathfrak{S}_{w(j+1)}$ meet the diagonal, or*
   - *neither of them meets the diagonal,*

   precisely as whether or not $\mathfrak{S}$ is truly orthogonal at j. And therefore $\cup_k \mathfrak{S}_{w(k)}$ has evenly many diagonal elements.

4. *No $\mathfrak{S}'_k$ intersects the diagonal. And therefore neither does $\cup_k \mathfrak{S}'_k$.*

The proposition will be proved below in §7.4.

Finally we are ready to define the image $(w, \mathfrak{S}')$ of $\mathfrak{S}$ under $\mathfrak{O}\pi$. We let w be the element of $I(d, 2d)$ associated to the distinguished subset $\cup_k \mathfrak{S}_{w(k)}$ of $\mathfrak{N}$; since $\cup_k \mathfrak{S}_{w(k)}$ is symmetric and has evenly many diagonal elements, it follows from Proposition 5.2.1 that w is in fact an element of $I(d)$. And we take $\mathfrak{S}' := \cup_k \mathfrak{S}'_k \cap \mathfrak{O}\mathfrak{N}$.

Remark 7.1.2 Setting

$$\pi(\mathfrak{S}_{j,j+1}) := (w_{j,j+1}, \mathfrak{S}'_{j,j+1}), \quad \mathfrak{S}' := \cup_{j \ \mathsf{odd}} \mathfrak{S}'_{j,j+1} \cap \mathfrak{O}\mathfrak{N},$$

and defining w to be the element of $I(d, 2d)$ associated to $\cup_{j \ \mathsf{odd}} \mathfrak{S}_{w_{j,j+1}}$ would give an equivalent definition of $\mathfrak{O}\pi$.

7.2 Illustration by an example

We illustrate the map $\mathfrak{O}\pi$ by means of an example. Let $d = 15$, and $v = (1, 2, 3, 4, 9, 10, 14, 16, 18, 19, 20, 23, 24, 25, 26)$. A monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$ is shown in Figure 7.2.1. Solid black dots indicate the elements that occur in $\mathfrak{S}$ with non-zero multiplicity. Integers written near the solid dots indicate multiplicities. The $\mathfrak{O}$-depth of $\mathfrak{S}$ is 5. The element $(21, 9)$ has $\mathfrak{O}$-depth 3 although it has depth 2 in $\mathfrak{S}$. Figure 7.2.2 shows the monomials $\mathfrak{S}^{\mathrm{pr}}_{1,2}$, $\mathfrak{S}^{\mathrm{pr}}_{3,4}$, and $\mathfrak{S}^{\mathrm{pr}}_{5,6}$. Solid dots, open dots, and crosses indicate elements of these monomials respectively. The monomial $\mathfrak{S}$ is truly orthogonal at 1 and 3 but not at 5: $\sigma_1 = (28, 2)$, $\sigma_3 = (21, 9)$, and $\sigma_5 = (15, 14)$.

Figure 7.2.3 shows the monomials $\mathfrak{S}_{1,2}$, $\mathfrak{S}_{3,4}$, and $\mathfrak{S}_{5,6}$ of $\mathfrak{N}$ and also their decomposition into blocks, and Figure 7.2.4 the monomials $\mathfrak{S}'_{1,2}$, $\mathfrak{S}'_{3,4}$, and $\mathfrak{S}'_{5,6}$.

We have

$$\mathfrak{S}_w = \{(15, 14), (17, 16), (21, 10), (7, 4), (27, 24), (28, 3), (30, 1), (29, 2)\}$$

hence $w = (7, 9, 15, 17, 18, 19, 20, 21, 23, 25, 26, 27, 28, 29, 30)$. It is easy to check that $w \in I(d)$. The monomial $\mathfrak{S}'$ is the intersection with $\mathfrak{O}\mathfrak{N}$ of the union of $\mathfrak{S}'_{1,2}$, $\mathfrak{S}'_{3,4}$, and $\mathfrak{S}'_{5,6}$—in other words it is just the monomial lying above $\mathfrak{d}$ in Figure 7.2.4.

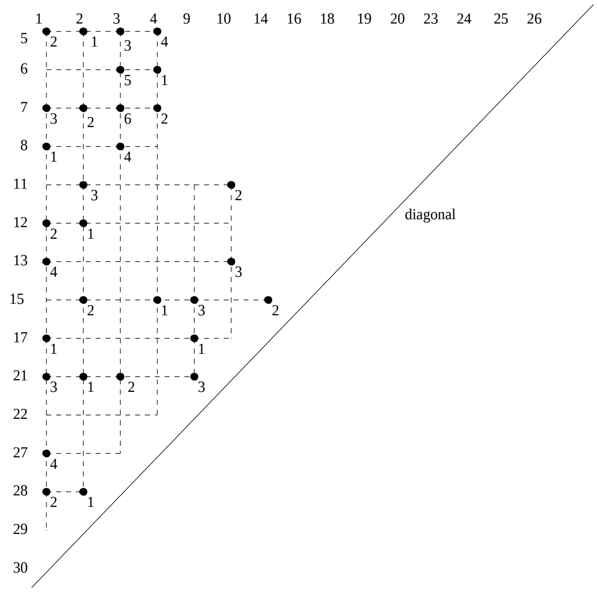


Figure 7.2.1: The monomial $\mathfrak{S}$

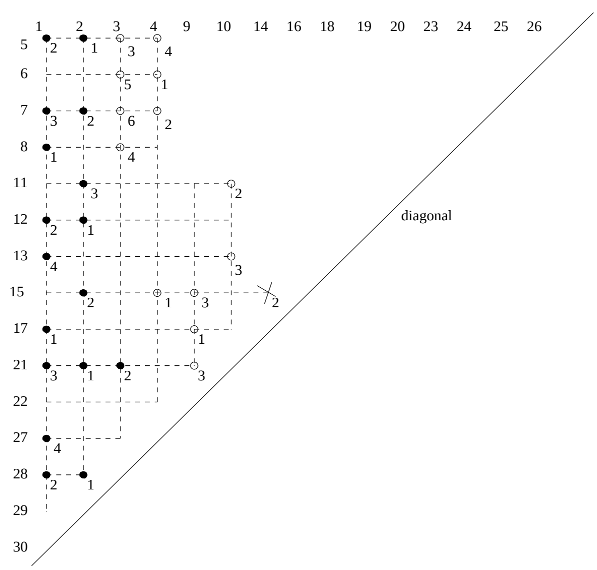


Figure 7.2.2: $\mathfrak{S}^{\mathrm{pr}}_{1,2}$, $\mathfrak{S}^{\mathrm{pr}}_{3,4}$, and $\mathfrak{S}^{\mathrm{pr}}_{5,6}$

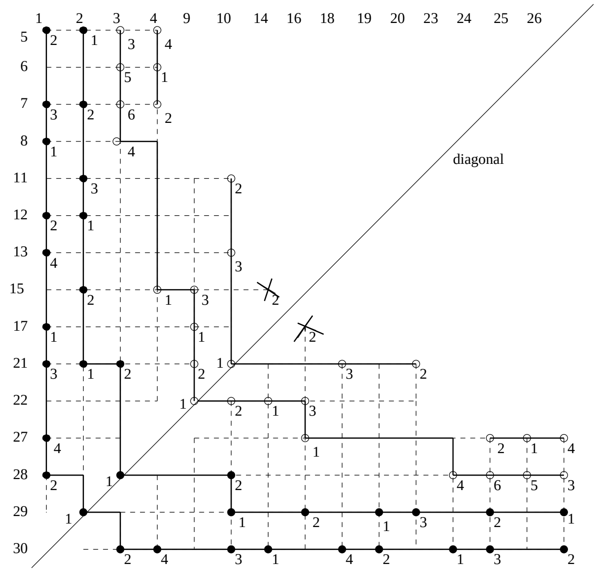


Figure 7.2.3: $\mathfrak{S}_{1,2}$, $\mathfrak{S}_{3,4}$, and $\mathfrak{S}_{5,6}$

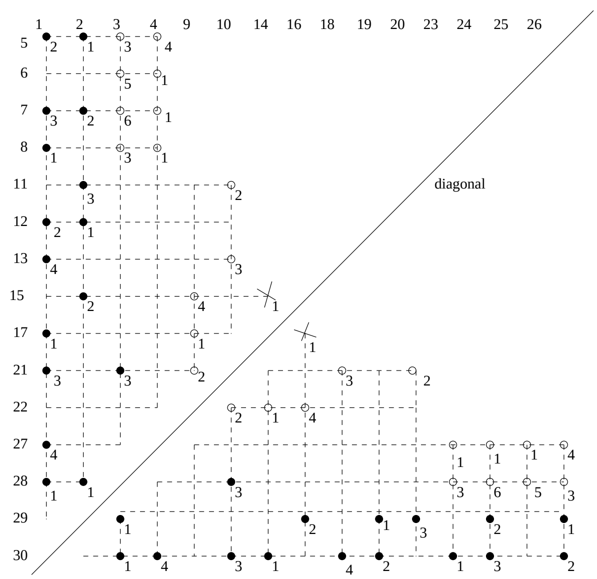


Figure 7.2.4: $\mathfrak{S}'_{1,2}$, $\mathfrak{S}'_{3,4}$, and $\mathfrak{S}'_{5,6}$

7.3 A proposition about $\mathfrak{S}_{j,j+1}$

The aim of this subsection is to show that $\mathfrak{S}_{j,j+1}$ has depth no more than 2—see item (1b) of Proposition 7.3.3. This basic fact was mentioned above in the description of $\mathfrak{O}\pi$ and is necessary (psychologically although not logically) to make sense of the definitions of $\mathfrak{S}_j$ and $\mathfrak{S}_{j+1}$. We prepare the way for Proposition 7.3.3 by way of two preliminary propositions. The first of these is about elements of $\mathfrak{O}$-depth j and $j+1$ in $\mathfrak{S}$, the second about the relation of these elements with σ_j.

Proposition 7.3.1 *1. $\mathfrak{S}_k^{\mathrm{pr}}$ has no comparable elements.*

2. For j an odd integer and β an element of $\mathfrak{S}_{j+1}^{\mathrm{pr}}$, there exists α in $\mathfrak{S}_j^{\mathrm{pr}}$ such that $\alpha > \beta$. In particular, the row index of σ_{j+1} (if σ_{j+1} exists) is less than the row index of σ_j.

Proof: (1) follows from Corollary 6.1.3 (4); (2) follows from Proposition 6.3.3 and Lemma 6.3.1 (5). $\Box$

Proposition 7.3.2 *Let j be an odd integer and let $\mathfrak{S}$ be truly orthogonal at j. Then*

1. *$p_v(\sigma_j) > p_h(\sigma_j)$; if $\alpha > p_v(\sigma_j)$, then $\alpha > \sigma_j$; if $\alpha > \sigma_j$, then $\alpha > p_h(\sigma_j)$.*
2. *No element of $\mathfrak{S}_j^{\mathrm{pr}}$ is comparable to $p_v(\sigma_j)$ or $p_h(\sigma_j)$.*
3. *No element of $\mathfrak{S}_{j+1}^{\mathrm{pr}}$ is comparable to $p_h(\sigma_j)$.*
4. *The following is not possible: $\alpha \in \mathfrak{S}_j^{\mathrm{pr}}$, $\beta \in \mathfrak{S}_{j+1}^{\mathrm{pr}}$, and $p_h(\alpha) > \beta$.*

Proof: (1) is trivial. (2) follows immediately from the definition of σ_j. We now prove (3). First suppose $\beta > p_h(\sigma_j)$ for some β in $\mathfrak{S}_{j+1}^{\mathrm{pr}}$. By (2) of Proposition 7.3.1, there exists α in $\mathfrak{S}_j^{\mathrm{pr}}$ such that $\alpha > \beta$. But then the row index of α exceeds that of σ_j, a contradiction to the choice of σ_j.

We claim that it is not possible for $\beta \in \mathfrak{S}_{j+1}^{\mathrm{pr}}$ to satisfy $p_h(\sigma_j) > \beta$. This being a special case of (4), we need only prove that statement. So suppose that α belongs to $\mathfrak{S}_j^{\mathrm{pr}}$ and that $p_h(\alpha) > \beta$. Let C be a v-chain in $\mathfrak{S}$ with tail α such that $\mathfrak{O}\text{-depth}_C(\alpha) = j$ (see Proposition 6.1.3 (1)). Concatenate C with $\alpha > \beta$ and call the resulting v-chain D. Then, by Lemma 6.3.1 (4), α is of type H in D, so that, by Lemma 6.3.1 (1), we have $\mathfrak{O}\text{-depth}_D(\beta) = \mathfrak{O}\text{-depth}_D(\alpha) + 2$. But, by Proposition 6.1.1 (2), $\mathfrak{O}\text{-depth}_D(\alpha) = \mathfrak{O}\text{-depth}_C(\alpha) = j$, so that $\mathfrak{O}\text{-depth}_{\mathfrak{S}}(\beta) \geq j+2$, a contradiction. $\Box$

Let $\mathfrak{S}_{j,j+1}(\text{ext})$ denote the set—not multiset—defined by:

$$\mathfrak{S}_{j,j+1}(\text{ext}) := \begin{cases} \mathfrak{S}_{j,j+1} \cup \{\sigma_j, \sigma_j^{\#}\} & \text{if } \mathfrak{S} \text{ is truly orthogonal at } j \\ \mathfrak{S}_{j,j+1} & \text{otherwise} \end{cases}$$

Here $\mathfrak{S}_{j,j+1}$ on the right stands for the underlying set of the multiset $\mathfrak{S}_{j,j+1}$ defined above. The set $\mathfrak{S}_{j,j+1}(\mathrm{ext})$ is the disjoint union of the sets $\mathfrak{S}_j(\mathrm{ext})$ and $\mathfrak{S}_{j+1}(\mathrm{ext})$ defined as follows (here again the terms on the right hand side denote the underlying sets of the corresponding multisets):

$$\mathfrak{S}_j(\mathrm{ext}) := \begin{cases} \mathfrak{S}_j^{\mathrm{pr}} \cup \left(\mathfrak{S}_j^{\mathrm{pr}}\right)^{\#} \cup \{p_v(\sigma)\} & \text{if } \mathfrak{S} \text{ is truly orthogonal at } j \\ \mathfrak{S}_j^{\mathrm{pr}} \cup \left(\mathfrak{S}_j^{\mathrm{pr}}\right)^{\#} & \text{otherwise} \end{cases}$$

$$\mathfrak{S}_{j+1}(\mathrm{ext}) := \begin{cases} \mathfrak{S}_{j+1}^{\mathrm{pr}} \cup \left(\mathfrak{S}_{j+1}^{\mathrm{pr}}\right)^{\#} \cup \{p_h(\sigma)\} & \text{if } \mathfrak{S} \text{ is truly orthogonal at } j \\ \mathfrak{S}_{j+1}^{\mathrm{pr}} \cup \left(\mathfrak{S}_{j+1}^{\mathrm{pr}}\right)^{\#} & \text{otherwise} \end{cases}$$

Proposition 7.3.3 *1. $\mathfrak{S}_j(\mathrm{ext})$ (respectively $\mathfrak{S}_{j+1}(\mathrm{ext})$) is precisely the set of elements of depth 1 (respectively 2) in $\mathfrak{S}_{j,j+1}(\mathrm{ext})$. In particular,*

(a) Neither $\mathfrak{S}_j(\mathrm{ext})$ nor $\mathfrak{S}_{j+1}(\mathrm{ext})$ contains comparable elements.

(b) The length of a v-chain in $\mathfrak{S}_{j,j+1}(\mathrm{ext})$ is at most 2.

(c) There is a v-chain of length 2 in $\mathfrak{S}_{j,j+1}$ unless $\mathfrak{S}_{j+1}(\mathrm{ext})$ is empty.

2. Let k be a positive integer, not necessarily odd. If there is in $\mathfrak{S}$ an element of $\mathfrak{O}$-depth at least k, then $\mathfrak{S}_k(\mathrm{ext})$ is non-empty. The converse also holds except possibly if k is even and $\mathfrak{S}$ is truly orthogonal at $k-1$. In particular, if $\mathfrak{S}_k(\mathrm{ext})$ is non-empty, then there is an element of $\mathfrak{O}$-depth at least $k-1$.

Proof: (1): It is enough to show that every element of $\mathfrak{S}_j(\mathrm{ext})(\mathrm{up})$ (respectively $\mathfrak{S}_{j+1}(\mathrm{ext})(\mathrm{up})$) is of depth 1 (respectively 2) in $\mathfrak{S}_{j,j+1}(\mathrm{ext})(\mathrm{up})$, for

- $\alpha > \beta$ implies $\alpha(\mathrm{up}) > \beta(\mathrm{up})$ for elements α, β of $\mathfrak{N}$.
- $\mathfrak{S}_{j,j+1}(\mathrm{ext}) = \mathfrak{S}_j(\mathrm{ext}) \cup \mathfrak{S}_{j+1}(\mathrm{ext})$.
- $\mathfrak{S}_{j,j+1}(\mathrm{ext})$, $\mathfrak{S}_j(\mathrm{ext})$, and $\mathfrak{S}_{j+1}(\mathrm{ext})$ are symmetric.

In turn, it is enough to show the following:

(i) Every element of $\mathfrak{S}_j(\mathrm{ext})(\mathrm{up})$ has depth 1.

(ii) $\mathfrak{S}_{j+1}(\mathrm{ext})(\mathrm{up})$ has no comparable elements.

(iii) Every element of $\mathfrak{S}_{j+1}(\mathrm{ext})(\mathrm{up})$ has depth at least 2.

Item (i) follows from Proposition 7.3.1 and Proposition 7.3.2 (2); item (ii) from Proposition 7.3.1 (1) and Proposition 7.3.2 (3); item (iii) from Proposition 7.3.1 (2) and Proposition 7.3.2 (1).

(2): The first assertion follows from Lemma 6.3.1 (5): if k is odd there is an element of $\mathfrak{O}$-depth k in $\mathfrak{S}$; if k is even and there is no element of $\mathfrak{O}$-depth k in $\mathfrak{S}$, then there is in $\mathfrak{S}$ an element of $\mathfrak{O}$-depth $k-1$ and of type H, so $\mathfrak{S}$ is truly orthogonal at $k-1$. The second assertion is clear from the definition of $\mathfrak{S}_k(\mathrm{ext})$. $\square$

Corollary 7.3.4 *1. No element of $\mathfrak{S}_{j,j+1}$ has depth more than 2.*

2. $\mathfrak{S}_{j+1}(\mathrm{ext}) = \mathfrak{S}_{j+1}$ and $\mathfrak{S}_j(\mathrm{ext}) \cap \mathfrak{S}_{j,j+1} = \mathfrak{S}_j$ (as sets). In particular, $\mathfrak{S}_{j+1} = \mathfrak{S}_{j,j+1} \cap \mathfrak{S}_{j+1}(\mathrm{ext})$ and $\mathfrak{S}_j = \mathfrak{S}_{j,j+1} \cap \mathfrak{S}_j(\mathrm{ext})$ as multisets defined by the intersection of a multiset with a subset.

PROOF: (1): Since $\mathfrak{S}_{j,j+1} \subseteq \mathfrak{S}_{j,j+1}(\mathrm{ext})$ (as sets), this follows immediately from (1b) of the proposition above.

(2): Since the union of $\mathfrak{S}_{j+1}(\mathrm{ext})$ (which always is contained in $\mathfrak{S}_{j,j+1}$) and $\mathfrak{S}_j(\mathrm{ext}) \cap \mathfrak{S}_{j,j+1}$ is all of $\mathfrak{S}_{j,j+1}$, and since $\mathfrak{S}_j$, $\mathfrak{S}_{j+1}$ are disjoint, it it enough to show that $\mathfrak{S}_{j+1}(\mathrm{ext}) \subseteq \mathfrak{S}_{j+1}$ and $\mathfrak{S}_j(\mathrm{ext}) \cap \mathfrak{S}_{j,j+1} \subseteq \mathfrak{S}_j$.

Now, since elements of $\mathfrak{S}_j(\mathrm{ext})$ have depth 1 even in $\mathfrak{S}_{j,j+1}(\mathrm{ext})$ (by item (1) of the proposition above), it is immediate that $\mathfrak{S}_j(\mathrm{ext}) \cap \mathfrak{S}_{j,j+1} \subseteq \mathfrak{S}_j$. And it follows from the proof of item (iii) in the proof of item (1) of the proposition above that an element of $\mathfrak{S}_{j+1}(\mathrm{ext})$ has depth 2 even in $\mathfrak{S}_{j,j+1}$ (not just in $\mathfrak{S}_{j,j+1}(\mathrm{ext})$), so that $\mathfrak{S}_{j+1}(\mathrm{ext}) \subseteq \mathfrak{S}_{j+1}$. □

7.4 Proof of Proposition 7.1.1

(1) The monomials $\mathfrak{S}_{j,j+1}$ are clearly symmetric. Observe that α in $\mathfrak{S}_{j,j+1}$ has the same depth as $\alpha^{\#}$, for $\alpha_1 > \alpha_2$ implies $\alpha(\mathrm{up}) > \alpha_2(\mathrm{up})$ and $\alpha(\mathrm{down}) > \alpha_2(\mathrm{down})$ for α_1, α_2 in $\mathfrak{N}$. Thus the monomials $\mathfrak{S}_k$ are symmetric. Since the map π of [7] respects #—see Proposition 5.7 of [4]—it follows that $\mathfrak{S}_{w(k)}$ and $\mathfrak{S}'_k$ are symmetric. Therefore so are $\cup_k \mathfrak{S}_{w(k)}$ and $\cup_k \mathfrak{S}'_k$.

(2) This follows from Corollary 9.3.6.

(3) If $\mathfrak{S}$ is truly orthogonal at j, then $p_v(\sigma_j)$ and $p_h(\sigma_j)$ are diagonal elements respectively in $\mathfrak{S}_j$ and $\mathfrak{S}_{j+1}$—see Corollary 7.3.4 (2). Thus both $\mathfrak{S}_j$ and $\mathfrak{S}_{j+1}$ have diagonal blocks in the sense of Proposition 5.10 (A) of [4]. It follows from the result just quoted that both $\mathfrak{S}_{w(j)}$ and $\mathfrak{S}_{w(j+1)}$ meet the diagonal. It is of course clear that each $\mathfrak{S}_{w(k)}$ meets the diagonal at most once since diagonal elements are clearly comparable but elements of $\mathfrak{S}_{w(k)}$ are not by Lemma 4.9 of [7].

Suppose that $\mathfrak{S}$ is not truly orthogonal at j. Then σ_j and $\sigma_j^{\#}$ belong to different blocks—this is equivalent to the definition of $\mathfrak{S}$ being not truly orthogonal at j. By Proposition 7.3.1 (2), it follows that σ_{j+1} and $\sigma_{j+1}^{\#}$ also belong to different blocks. So neither $\mathfrak{S}_j$ nor $\mathfrak{S}_{j+1}$ has a diagonal block.

(4) If $\mathfrak{S}$ is not truly orthogonal at j, then neither $\mathfrak{S}_j$ nor $\mathfrak{S}_{j+1}$ has a diagonal block (as has just been said above), and it follows from Proposition 5.10 (A) of [4] that neither $\mathfrak{S}'_j$ nor $\mathfrak{S}'_{j+1}$ meets the diagonal.

So suppose that $\mathfrak{S}$ is truly orthogonal at j. Then both $\mathfrak{S}_j$ and $\mathfrak{S}_{j+1}$ have a diagonal entry each of multiplicity 1, namely $p_v(\sigma_j)$ and $p_h(\sigma_j)$ respectively. It is clear from the definition of σ_j that no element of $\mathfrak{S}_j(\mathrm{up})$ shares its row index with $p_v(\sigma_j)$. And it follows from Proposition 7.3.1 (2) that no element of $\mathfrak{S}_{j+1}(\mathrm{up})$ shares its row index with $p_h(\sigma_j)$. It now follows from the proof of Proposition 5.10 (B) of [4]—see the last line of that proof—that neither $\mathfrak{S}'_j$ nor $\mathfrak{S}'_{j+1}$ meets the diagonal. □

7.5 More observations

Proposition 7.5.1 *The length of any v-chain in $\mathfrak{S}_{j,j+1} \cup \mathfrak{S}'_j \cup \mathfrak{S}'_{j+1}$ is at most* 2.

Proof: By Corollary 7.3.4 (1), the length of any v-chain in $\mathfrak{S}_{j,j+1}$ is at most 2. Applying Lemma 9.1.1 to $\mathfrak{S}_{j,j+1}$, we get the desired result. □

Proposition 7.5.2 1. *For an element $\alpha' = (r, c)$ of $\mathfrak{S}'_k(\mathrm{up})$, there exists an element $\alpha = (r, C)$ of $\mathfrak{S}_k^{\mathrm{pr}}$ with $C \leq c$.*

2. *For an element $\alpha' = (r, c)$ of $\mathfrak{S}'_{j+1}(\mathrm{up})$, there exists an element $\alpha = (R, c)$ of $\mathfrak{S}_{j+1}(\mathrm{up})$ with $r \leq R$.*

3. *For an element α' of $\mathfrak{S}'_{j+1}(\mathrm{up})$, there exists an element α of $\mathfrak{S}_j^{\mathrm{pr}}$ with $\alpha > \alpha'$.*

Proof: (1) That there exists α in $\mathfrak{S}_k(\mathrm{up})$ with $C \leq c$ follows from the definition of $\mathfrak{S}'_k(\mathrm{up})$. Clearly such an α cannot be on the diagonal, so α belongs to $\mathfrak{S}_k^{\mathrm{pr}}$.

(2) As in the proof of (1), it follows from the definition of $\mathfrak{S}'_{j+1}$ that there exists $\alpha = (R, c)$ in $\mathfrak{S}_{j+1}$ with $r \leq R$. If α lies strictly below the diagonal, then $c > R^*$, so that $\alpha^* = (c^*, R^*) > \alpha' = (r, c)$, a contradiction to Lemma 9.1.1 (α^* belongs to $\mathfrak{S}_{j+1}$ by the symmetry of $\mathfrak{S}_{j+1}$). Thus α belongs to $\mathfrak{S}_{j+1}(\mathrm{up})$.

(3) Writing $\alpha' = (r, c)$, by (1), we can find an $\beta = (r, C)$ in $\mathfrak{S}_{j+1}^{\mathrm{pr}}$ with $C \leq c$. By Proposition 7.3.1 (2), there exists α in $\mathfrak{S}_{j,j+1}^{\mathrm{pr}}$ such that $\alpha > \beta$. □

Corollary 7.5.3 *If in $\mathfrak{S}'_j(\mathrm{up}) \cup \mathfrak{S}'_{j+1}(\mathrm{up})$ there exists an element with horizontal projection in $\mathfrak{N}$, then $\mathfrak{S}$ is truly orthogonal at j.*

Proof: Follows directly from Proposition 7.5.2 (1) and (3). □

Proposition 7.5.4 *The $\mathfrak{O}$-depth of an element in $\mathfrak{S}_j^{\mathrm{pr}} \cup \mathfrak{S}_{j+1}^{\mathrm{pr}}$ is at most* 2. *More strongly, the $\mathfrak{O}$-depth of an element in $\mathfrak{S}_{j,j+1}^{\mathrm{pr}} \cup \mathfrak{S}'_j(\mathrm{up}) \cup \mathfrak{S}'_{j+1}(\mathrm{up})$ is at most* 2.

Proof: It is enough to show that no element in $\mathfrak{S}'_j(\mathrm{up}) \cup \mathfrak{S}'_{j+1}(\mathrm{up})$ has $\mathfrak{O}$-depth more than 2, for we may assume by increasing multiplicities that $\mathfrak{S}_j^{\mathrm{pr}} \subseteq \mathfrak{S}'_j(\mathrm{up})$ and $\mathfrak{S}_{j+1}^{\mathrm{pr}} \subseteq \mathfrak{S}'_{j+1}(\mathrm{up})$ (as sets). It follows from Proposition 7.5.1 that a v-chain in $\mathfrak{S}'_j(\mathrm{up}) \cup \mathfrak{S}'_{j+1}(\mathrm{up})$ has length at most 2. Let $\alpha'_1 = (r_1, c_1) > \alpha'_2 = (r_2, c_2)$ be such a v-chain. It follows from the proof of Corollary 4.14 (2) of [7] that $\alpha'_1 \in \mathfrak{S}'_j(\mathrm{up})$ and $\alpha'_2 \in \mathfrak{S}'_{j+1}(\mathrm{up})$. By item (1) of Lemma 6.3.1, it is enough to rule out the following possibility: α'_1 is of type H in $\alpha'_1 > \alpha'_2$ and $p_h(\alpha'_1) > \alpha'_2$.

Suppose that this is the case. By Proposition 7.5.2 (1) and (2), it follows that there exist elements $\alpha_1 = (r_1, C_1) \in \mathfrak{S}_{j,j+1}^{\mathrm{pr}}$ and $\alpha_2 = (R_2, c_2) \in \mathfrak{S}_{j+1}(\mathrm{up})$ with $C_1 \leq c_1$ and $r_2 \leq R_2$. Since $p_h(\alpha'_1) > \alpha'_2$, it follows that $\alpha_1 > \alpha_2$. Now, if $\alpha_2 = p_h(\sigma_j)$, then Proposition 7.3.2 (2) is contradicted; if α_2 belongs to $\mathfrak{S}_{j+1}^{\mathrm{pr}}$, Proposition 7.3.2 (4) is contradicted (because $p_h(\alpha_1) > \alpha_2$). □

8 The map $\mathfrak{O}\phi$

The purpose of this section is to describe the map $\mathfrak{O}\phi$ and prove some basic facts about it. Certain proofs here refer to results from §9, but there is no circularity—to postpone the definition of $\mathfrak{O}\phi$ until all the results needed for it have been proved would hurt rather than help readability. As in §7, the symbol j will be reserved for an odd integer throughout this section.

8.1 Description of $\mathfrak{O}\phi$

The map $\mathfrak{O}\phi$ takes as input a pair $(w, \mathfrak{T})$, where $\mathfrak{T}$ is a monomial, possibly empty, in $\mathfrak{O}\mathfrak{N}$ and $w \geq v$ an element of $I(d)$ that $\mathfrak{O}$-dominates $\mathfrak{T}$, and produces as output a monomial $\mathfrak{T}^*$ of $\mathfrak{O}\mathfrak{N}$. To describe $\mathfrak{O}\phi$, we first partition $\mathfrak{T}$ into subsets $\mathfrak{T}_{w,j,j+1}$. As the subscript w in $\mathfrak{T}_{w,j,j+1}$ suggests, this partition depends on w.

For an odd integer j, let $\mathfrak{S}_w^j$ (respectively $\mathfrak{S}_{w,j,j+1}$) denote the subset of $\mathfrak{S}_w$ consisting of those elements that are j-deep (respectively that are j deep but not $j+2$ deep, or equivalently of depth j or $j+1$) in $\mathfrak{S}_w$ in the sense of [7, §4]. Since $\mathfrak{S}_w$ is distinguished, symmetric, and has evenly many elements on the diagonal $\mathfrak{d}$, it follows that $\mathfrak{S}_w^j$ and $\mathfrak{S}_{w,j,j+1}$ too have these properties, and that, in fact, the number of diagonal elements of $\mathfrak{S}_{w,j,j+1}$ is either 0 or 2 (in the latter case, the elements have to be distinct since $\mathfrak{S}_w$ is distinguished and so is multiplicity free). Let us denote by w^j and $w_{j,j+1}$ the elements of $I(d)$ corresponding to $\mathfrak{S}_w^j$ and $\mathfrak{S}_{w,j,j+1}$ by Proposition 5.2.1.

Let $\mathfrak{T}_{w,j,j+1}$ denote the subset of $\mathfrak{T}$ consisting of those elements α such that

- every v-chain in $\mathfrak{T}$ with head α is $\mathfrak{O}$-dominated by w^j, and
- there exists a v-chain in $\mathfrak{T}$ with head α that is not $\mathfrak{O}$-dominated by w^{j+2}.

It is evident that the subsets $\mathfrak{T}_{w,j,j+1}$ are disjoint (as j varies over the odd integers) and that their union is all of $\mathfrak{T}$ (for $w = w^1$ $\mathfrak{O}$-dominates all v-chains in $\mathfrak{T}$ by hypothesis and $\mathfrak{S}_w^j$ is empty for large j and so $w^j = v$). In other words, the $\mathfrak{T}_{w,j,j+1}$ form a partition of $\mathfrak{T}$.

Lemma 8.1.1 *1. The length of a v-chain in $\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1}$ is at most 2. In fact, the $\mathfrak{O}$-depth of any element in $\mathfrak{T}_{w,j,j+1}$ is at most 2.*

2. $w_{j,j+1}$ $\mathfrak{O}$-dominates $\mathfrak{T}_{w,j,j+1}$.

PROOF: The lemma follows rather easily from Corollary 9.2.3 as we now show. Let C be a v-chain in $\mathfrak{T}_{w,j,j+1}$. Let τ be the tail of C. Choose a v-chain D in $\mathfrak{T}$ with head τ that is not $\mathfrak{O}$-dominated by w^{j+2}. Let E be the concatenation of C with D. Since the head of E belongs to $\mathfrak{T}_{w,j,j+1}$, it follows that E is $\mathfrak{O}$-dominated by w^j. It follows from (the only if part of) Corollary 9.2.3 (applied with $\mathfrak{S} = E$ and $x = w^j$) that $w_{j,j+1}$ $\mathfrak{O}$-dominates $E_1^{\mathrm{pr}} \cup E_2^{\mathrm{pr}}$ and w^{j+2} $\mathfrak{O}$-dominates $E^{3,\mathrm{pr}}$. This means $\tau \notin E^{3,\mathrm{pr}}$, so $\tau \in E_1^{\mathrm{pr}} \cup E_2^{\mathrm{pr}}$, and so $C \subseteq E_1^{\mathrm{pr}} \cup E_2^{\mathrm{pr}}$. This proves (2). By Proposition 6.1.1 (2), the $\mathfrak{O}$-depths of

elements of C are the same in C and E, so $C \subseteq C_1^{\mathrm{pr}} \cup C_2^{\mathrm{pr}}$, which proves the second assertion of (1). The first assertion of (1) follows from the second (see Lemma 6.2.1). $\square$

Corollary 8.1.2 *$w_{j,j+1}$ dominates $\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1}$ in the sense of [7].*

PROOF: This follows from (2) of Lemma 8.1.1 and Corollary 5.3.6 (the latter applied with $\mathfrak{S} = \mathfrak{T}_{w,j,j+1}$ and $w = w_{j,j+1}$). $\square$

We may therefore apply the map ϕ of [7, §4] to the pair $(w_{j,j+1}, \mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})$ to obtain a monomial $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^{\star}$ in $\mathfrak{N}$. In applying ϕ, there is the partitioning of $\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1}$ into "pieces", these being indexed by elements of $\mathfrak{S}_{w_{j,j+1}} = \mathfrak{S}_{w,j,j+1}$—observe that the elements of depth 1 (respectively 2) of $\mathfrak{S}_{w,j,j+1}$ are precisely those of $\mathfrak{S}_w$ of depth j (respectively $j+1$). We denote by $\mathfrak{P}_\beta$ the piece of $\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1}$ corresponding to β in $\mathfrak{S}_{w_{j,j+1}}$. We also use the notation $\mathfrak{P}^*_\beta$ as in [7]. Moreover, we will use the phrase *piece of* $\mathfrak{T}$ (with respect to w being implicitly understood) to refer to a piece of $\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1}$ for some odd integer j.

> Caution: Thinking of $\mathfrak{T}$ as a monomial in $\mathfrak{N}$ and w as an element of $I(d, 2d)$ that dominates it, there is, as in [7], the notion of "piece of $\mathfrak{T}$" (with respect to w). The two notions of "piece" are different.

Lemma 8.1.3

1. *The monomial $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^{\star}$ is symmetric and has either none or two distinct diagonal elements depending exactly on whether $\mathfrak{S}_{w_{j,j+1}} = \mathfrak{S}_{w,j,j+1}$ has 0 or 2 elements on the diagonal.*
2. *The depth of $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^{\star}$ is 2; and $\cup_{\beta \in (\mathfrak{S}_w)_j} \mathfrak{P}^*_\beta$, $\cup_{\beta \in (\mathfrak{S}_w)_{j+1}} \mathfrak{P}^*_\beta$ are respectively the elements of depth 1 and 2 in $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^{\star}$.*

PROOF: (1) The symmetry follows by combining Proposition 5.6 of [4], which says that the map π respects the involution $\#$, with Proposition 4.2 of [7], which says that π and ϕ are are inverses of each other.

The assertion about diagonal elements follows by combining item (B) of [4, Proposition 5.10], which is an assertion about the existence and relative multiplicities of diagonal elements in $\mathfrak{B}$ and $\mathfrak{B}'$ where $\mathfrak{B}$ is a diagonal block of a monomial in $\mathfrak{N}$, and Proposition 4.2 of [7].

(2) It follows from Propositions 4.2 of [7] that the map π (described in §4 of that paper) applied to $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^{\star}$ results in the pair $(w_{j,j+1}, \mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}_{w,j,j+1})$. It now follows from Lemma 4.16 of [7] that the depth of $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^{\star}$ is exactly 2. The latter assertions again follow from the results of [7]—in fact, the proof that $\pi \circ \phi$ is identity on pages 47–49 of [7] shows that the $\mathfrak{P}^*_\beta$ are the blocks in the sense of [7] of the monomial $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^{\star}$. $\square$

Suppose that $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^{\star}$ contains the pair (a, a^*), (b, b^*) of diagonal elements with $a > b$. We call the pair (b, a^*), (a, b^*) the "twists," and

set $\delta_j := (b, a^*)$. In other words, δ_j is the element of the twisted pair that lies above the diagonal—observe that the twisted elements are reflections of each other. We allow ourselves the following ways of expressing the condition that $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^\star$ has diagonal elements: *δ_j exists; w is diagonal at j* (the latter expression is justified by the lemma above).

With notation as above, consider the new monomial defined as

$$\begin{cases} (\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^\star & \text{if } w \text{ is not diagonal at } j \\ \Big((\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^\star \setminus \mathfrak{d}\Big) \cup \{\delta_j, \delta_j^{\#}\} & \text{if } w \text{ is diagonal at } j \end{cases}$$

This new monomial is symmetric and contains no diagonal elements. Its intersection with $\mathfrak{O}\mathfrak{N}$ is denoted $\mathfrak{T}^\star_{w,j,j+1}$. In other words, $\mathfrak{T}^\star_{w,j,j+1}$ is the intersection of the new monomial with the subset of $\mathfrak{N}$ of those elements that lie strictly above the diagonal.

The union of $\mathfrak{T}^\star_{w,j,j+1}$ over all odd integers j is defined to be $\mathfrak{T}^\star_w$, the result of $\mathfrak{O}\phi$ applied to $(w, \mathfrak{T})$. This finishes the description of the map $\mathfrak{O}\phi$.

For β in $\mathfrak{S}_{w,j,j+1}(\text{up})$, we define the "orthogonal piece-star" $\mathfrak{O}\mathfrak{P}^*_\beta$ corresponding to β as

$$\mathfrak{O}\mathfrak{P}^*_\beta := \begin{cases} \mathfrak{P}^*_\beta = \mathfrak{P}^*_\beta(\text{up}) & \text{if } \beta \text{ is not on the diagonal} \\ \mathfrak{P}^*_\beta \cap \mathfrak{O}\mathfrak{N} & \text{if } \beta \in (\mathfrak{S}_w)_{j+1} \text{ is on the diagonal} \\ \{\mathfrak{P}^*_\beta \cap \mathfrak{O}\mathfrak{N}\} \cup \{\delta_j\} & \text{if } \beta \in (\mathfrak{S}_w)_j \text{ is on the diagonal} \end{cases} \tag{8.1.1}$$

With this, we can say that $\mathfrak{T}^\star_w$ is the union of $\mathfrak{O}\mathfrak{P}^*_\beta$ as β varies over $\mathfrak{S}_w(\text{up})$.

Lemma 8.1.4 *Suppose that $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^\star$ contains the pair (a, a^*), (b, b^*) of diagonal elements with $a > b$. Let*

$$\ldots,\ (r_1, c_1),\ (a, a^*),\ (c_1^*, r_1^*),\ \ldots; \qquad \ldots,\ (r_2, c_2),\ (b, b^*),\ (c_2^*, r_2^*),\ \ldots$$

be respectively the elements of depth 1 and 2 of $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^\star$ arranged in increasing order of row and column indices. Then

1. *$c_1 \leq a^*$ and $r_1 \leq b$ (assuming (r_1, c_1) exists); and*
2. *$r_2 < b$ and $c_2 \leq b^*$ (assuming (r_2, c_2) exists).*

PROOF: (1) Suppose that (r_1, c_1) exists. It is clear that $c_1 \leq a^*$. From way the map ϕ of [7] is defined, it follows that (r_1, a^*) is an element of $\mathfrak{T}_{w,j,j+1}$. Suppose that $r_1 > b$. Then $p_h(r_1, a^*) = (r_1, r_1^*)$ belongs to $\mathfrak{N}$. We consider two cases.

If (r_2, c_2) exists, then, again from the definition of the map ϕ, it follows that (r_2, b^*) is an element of $\mathfrak{T}_{w,j,j+1}$. But then $p_h(r_1, a^*) = (r_1, r_1^*) > (b, b^*)$ and (b, b^*) dominates (r_2, b^*), which means that the v-chain $(r_1, a^*) > (r_2, b^*)$ (note that $a^* < b^*$ because $a > b$ by hypothesis) in $\mathfrak{T}_{w,j,j+1}$ has $\mathfrak{O}$-depth more than 2, a contradiction to Lemma 8.1.1 (1).

Now suppose that (r_2, c_2) does not exist. (Then (b, b^*) is the diagonal element in $(\mathfrak{S}_w)_{j+1}$.) Consider the singleton v-chain $C := \{(r_1, a^*)\}$ in $\mathfrak{T}_{w,j,j+1}$.

Then $\mathfrak{S}_C = \{(a, a^*), (r_1, r_1^*)\}$ which is not dominated by $w_{j,j+1}$, a contradiction to Lemma 8.1.1 (2).

(2) Suppose that (r_2, c_2) exists. Then there exists, by the definition of the map ϕ, an element (r_2, b^*) in $\mathfrak{T}_{w,j,j+1}$. Since (r_2, b^*) lies above the diagonal, it follows that $r_2 < b$. That $c_2 \leq b^*$ is clear. $\square$

8.2 Basic facts about $\mathfrak{T}_{w,j,j+1}$ and $\mathfrak{T}^\star_{w,j,j+1}$

Lemma 8.2.1 *1. Let $\alpha' > \alpha$ be elements of $\mathfrak{T}$. Let j and j' be the odd integers such that $\alpha' \in \mathfrak{T}_{w,j',j'+1}$ and $\alpha \in \mathfrak{T}_{w,j,j+1}$. Then $j' \leq j$.*

2. If, further, either

(a) there exists μ in $\mathfrak{T}$ such that $\alpha' > \mu > \alpha$, or

(b) $\alpha' \in \mathfrak{P}_{\beta'}$ for β' in $(\mathfrak{S}_w)_{j'+1}$,

then $j' < j$.

Proof: (1) By hypothesis, every v-chain with head α' is $\mathfrak{O}$-dominated by $w^{j'}$. This implies, by Corollary 6.1.2, that every v-chain with head α is $\mathfrak{O}$-dominated by $w^{j'}$. This shows $j' \leq j$.

(2a) Suppose that $j' = j$. It follows from (1) that α', μ, and α all belong to $\mathfrak{T}_{w,j,j+1}$. But then $\alpha' > \mu > \alpha$ is a v-chain of length 3 in $\mathfrak{T}_{w,j,j+1}$, a contradiction to Lemma 8.1.1 (1).

(2b) Suppose that $j' = j$. Then $\alpha' > \alpha$ is a v-chain in $\mathfrak{T}_{w,j,j+1}$. Being of length 2, it cannot be dominated by $(\mathfrak{S}_w)_{j+1}$, which means, by the definition of $\mathfrak{P}_{\beta'}$, that α' cannot belong to $\mathfrak{P}_{\beta'}$, a contradiction. $\square$

Proposition 8.2.2 *1. The length of a v-chain in $\mathfrak{T}^\star_{w,j,j+1}$ is at most 2.*

2. The $\mathfrak{O}$-depth of $\mathfrak{T}^\star_{w,j,j+1}$ is at most 2.

3. $\cup_{\beta \in (\mathfrak{S}_w)_j(\mathrm{up})} \mathfrak{O}\mathfrak{P}^*_\beta$ is precisely the set of depth 1 elements of $\mathfrak{T}^\star_{w,j,j+1}$ (in particular, no two elements there are comparable); if δ_j exists, then it is the last element of $\cup_{\beta \in (\mathfrak{S}_w)_j(\mathrm{up})} \mathfrak{O}\mathfrak{P}^*_\beta$ when the elements are arranged in increasing order of row and column indices.

4. $\cup_{\beta \in (\mathfrak{S}_w)_{j+1}(\mathrm{up})} \mathfrak{O}\mathfrak{P}^*_\beta$ is precisely the set of depth 2 elements of $\mathfrak{T}^\star_{w,j,j+1}$ (in particular, no two elements there are comparable); if δ_j exists, then its row index exceeds the row index of any element in $\cup_{\beta \in (\mathfrak{S}_w)_{j+1}(\mathrm{up})} \mathfrak{O}\mathfrak{P}^*_\beta$.

Proof: For (1), it is enough, given Lemma 8.1.3 (2), to show that δ_j is not comparable to any element of depth 1 of $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^\#_{w,j,j+1})^\star$, and this follows from Lemma 8.1.4 (1). In fact, the above argument proves also (3).

For (4), it is enough, given Lemma 8.1.3 (2), the symmetry of the monomials involved in that lemma, and the observation that $\alpha > \beta$ implies $\alpha(\mathrm{up}) >$

β(up) for elements α, β of $\mathfrak{N}$, to show the following: if $(a, a^*) > \gamma = (e, f)$ for γ an element of $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^\star$ lying (strictly) above the diagonal, then $\delta_j > \gamma$. But this follows from Lemma 8.1.4 (2): γ is a depth 2 element in $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^{\#}_{w,j,j+1})^\star$, and we have $e \leq r_2 < b$ (and $a^* < f$ since $(a, a^*) > \gamma$). In fact, the above argument proves also (2): observe that $f \leq b^*$ (Lemma 8.1.4 (2)). □

9 Some Lemmas

The main combinatorial results of this paper are Propositions 4.1.1 and 4.1.2. They are analogues respectively of Propositions 4.1 and 4.2 of [7]. We have tried to preserve the structure of the proofs in [7] of those propositions. The proofs in [7] rely on certain lemmas and it is natural therefore to first establish the orthogonal analogues of those. The purpose of this section is precisely that. Needless to say that the lemmas (especially those in §9.4) may be unintelligible until one tries to read §10.

The division of this section into four subsections is also suggested by the structure of the proofs in [7]. Each subsection has at its beginning a brief description of its contents.

9.1 Lemmas from the Grassmannian case

In this subsection, the terminology and notation of [7, §4] are in force. The statements here could have been made in [7, §4] and would perhaps have improved the efficiency of the proofs there, but do not appear there explicitly.

Let $\mathfrak{S}$ be a monomial in $\mathfrak{N}$. Recall from [7] the notion of *depth* of an element α in $\mathfrak{S}$: it is the largest possible length of a v-chain in $\mathfrak{S}$ with tail α and denoted $\text{depth}_{\mathfrak{S}}\alpha$. The *depth* of $\mathfrak{S}$ is the maximum of the depths in it of all its elements. We denote by $\mathfrak{S}_k$ the set of elements of depth k of $\mathfrak{S}$ (as in [7]) and by $\mathfrak{S}^k$ the set of elements of depth at least k of $\mathfrak{S}$.

> Caution: For a monomial $\mathfrak{S}$ of $\mathfrak{O}\mathfrak{N}$, we have introduced in §7.1 the notation $\mathfrak{S}_k$. That is different from the $\mathfrak{S}_k$ we have just defined.

Lemma 9.1.1 *Let $\mathfrak{S}$ be a monomial in $\mathfrak{N}$, and let $\pi(\mathfrak{S}) = (w, \mathfrak{S}')$, where π is the map defined in [7, §4]. Then the maximum length of a v-chain in $\mathfrak{S} \cup \mathfrak{S}'$ is the same as the maximum length of a v-chain in $\mathfrak{S}$.*

Proof: We use the notation of [7, §4] freely. Let d be the maximum length of a v-chain in $\mathfrak{S}$. Suppose $\alpha_1 > \ldots > \alpha_\ell$ is a v-chain in $\mathfrak{S} \cup \mathfrak{S}'$. Let $i_1, \ldots, i_\ell$ be such that α_j belongs to $\mathfrak{S}_{i_j} \cup \mathfrak{S}'_{i_j}$ (the integers i_j are uniquely determined—see Corollary 5.4 of [4]). We claim that $i_1 < \ldots < i_\ell$. This suffices to prove the lemma, for $\mathfrak{S}_k \cup \mathfrak{S}'_k$ is empty for $k > d$.

To prove the claim, it is enough to show $i_1 < i_2$. It follows from Lemma 4.10 of [7] that $i_1 \neq i_2$. We now assume that $i_1 > i_2$ and arrive at a contradiction.

First suppose that $\alpha_1 \in \mathfrak{S}_{i_1}$. Then, by the definition of $\mathfrak{S}_{i_1}$, there exists β in $\mathfrak{S}_{i_2}$ with $\beta > \alpha_1$. Now $\beta > \alpha_2$ and both β, α_2 belong to $\mathfrak{S}_{i_2} \cup \mathfrak{S}'_{i_2}$, a contradiction to [7, Lemma 4.10]. If $\alpha_1 = (r, c)$ belongs to $\mathfrak{S}'_{i_1}$, then, by the definition of $\mathfrak{S}'_{i_1}$, there exists (r, a) in $\mathfrak{S}_{i_1}$ with $a \leq c$, and there exists β in $\mathfrak{S}_{i_2}$ with $\beta > (r, a)$. This leads to the same contradiction as before. $\square$

Lemma 9.1.2 *Let $\mathfrak{B}$ and $\mathfrak{U}$ be monomials in $\mathfrak{N}$. Assume that*

- *the elements of $\mathfrak{B}$ form a single block (in the sense of [7, Page 38]).*
- *$\mathfrak{U}$ has depth 1 (equivalently, there are no comparable elements in $\mathfrak{U}$).*
- *for every $\beta = (r, c)$ in $\mathfrak{B}$, there exist $\gamma^1(\beta) = (R^1, C^1)$, and $\gamma^2(\beta) = (R^2, C^2)$ in $\mathfrak{U}$ such that*

$$C^1 < c, \quad C^2 < R^1, \quad r < R^2$$

  (this holds, for example, when there exists $\gamma(\beta)$ in $\mathfrak{U}$ such that $\gamma(\beta) > \beta$: take $\gamma^1(\beta) = \gamma^2(\beta) = \gamma(\beta)$).

Then there exists a unique block $\mathfrak{C}$ of $\mathfrak{U}$ such that $w(\mathfrak{C}) > w(\mathfrak{B})$.

Proof: It is useful to isolate the following observation:

Lemma 9.1.3 *Let (r_1, c_1) and (r_2, c_2) be elements of $\mathfrak{N}$ with $c_2 < r_1 \leq r_2$. Let $\gamma_1^1 = (R_1^1, C_1^1)$, $\gamma_1^2 = (R_1^2, C_1^2)$ and $\gamma_2^1 = (R_2^1, C_2^1)$, $\gamma_2^2 = (R_2^2, C_2^2)$ be elements of $\mathfrak{N}$ such that*

1. *$C_1^1 \leq c_1, \quad C_1^2 < R_1^1, \quad r_1 \leq R_1^2$.*
2. *$C_2^1 \leq c_2, \quad C_2^2 < R_2^1, \quad r_2 \leq R_2^2$.*
3. *No two of γ_1^1, γ_1^2, γ_2^1, γ_2^2 are comparable (they could well be equal and this is important for us—see our definition of comparability).*

Then the monomial $\{\gamma_1^1, \gamma_1^2, \gamma_2^1, \gamma_2^2\}$ consists of a single block.

Proof: It follows from assumption (1) that γ_1^1 and γ_1^2 belong to a single block:

- if $R_1^1 < R_1^2$, then $C_1^2 < R_1^1$ becomes relevant;
- if $R_1^2 < R_1^1$, then the other two inequalities in (1) become relevant: $C_1^1 \leq c_1 < r_1 \leq R_1^2$.

Similarly it follows from assumption (2) that γ_2^1 and γ_2^2 belong to a single block.

We therefore need only consider the cases when, in the arrangement of the elements $\{\gamma_1^1, \gamma_1^2, \gamma_2^1, \gamma_2^2\}$ in increasing order of row indices, both γ_1^1, γ_1^2 come before or after γ_2^1, γ_2^2. In the former case, the first sequence of inequalities below shows that γ_1^2 and γ_2^1 belong to the same block, and we are done; in the latter case, the second sequence of inequalities below shows that γ_2^2 and γ_1^1 belong to the same block, and we are done:

- $C_2^1 \leq c_2 < r_1 \leq R_1^2$.
- $C_1^1 \leq c_1 < r_1 \leq r_2 \leq R_2^2$. □

Continuing with the proof of Lemma 9.1.2, we first prove the existence part. Arrange the elements of $\mathfrak{B}$ in non-decreasing order of row numbers as well as column numbers (this is possible since there are no comparable elements in $\mathfrak{B}$). If $\beta_1 = (r_1, c_1)$ and $\beta_2 = (r_2, c_2)$ are successive elements, then $c_2 < r_1 \leq r_2$ (since $\mathfrak{B}$ is a single block). Apply Lemma 9.1.3 with $\gamma_1^1 = \gamma^1(\beta_1)$, $\gamma_1^2 = \gamma^2(\beta_1)$, and $\gamma_2^1 = \gamma^1(\beta_2)$, $\gamma_2^2 = \gamma^1(\beta_2)$. We conclude that $\{\gamma_1^1, \gamma_1^2, \gamma_2^1, \gamma_2^2\}$ belongs to a single block, say $\mathfrak{C}$, of $\mathfrak{U}$. Continuing thus, we conclude that all $\gamma^1(\beta)$ and $\gamma^2(\beta)$, as β varies over $\mathfrak{B}$, belong to $\mathfrak{C}$. Since the row (respectively column) index of $w(\mathfrak{C})$ is the maximum (respectively minimum) of all row (respectively column) indices of elements of $\mathfrak{C}$ (and similarly for $\mathfrak{B}$), it follows that $w(\mathfrak{C}) > w(\mathfrak{B})$.

To prove uniqueness, let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two blocks of $\mathfrak{U}$ with $w(\mathfrak{C}_1) > w(\mathfrak{B})$ and $w(\mathfrak{C}_2) > w(\mathfrak{B})$. Apply the lemma with $(r_1, c_1) = (r_2, c_2) = w(\mathfrak{B})$ and $\gamma_1^1 = \gamma_1^2 = w(\mathfrak{C}_1)$ and $\gamma_2^1 = \gamma_2^2 = w(\mathfrak{C}_2)$; it follows from [7, Lemma 4.9] that $w(\mathfrak{C}_1)$ and $w(\mathfrak{C}_2)$ are not comparable. But, unless $\mathfrak{C}_1 = \mathfrak{C}_2$, neither is the monomial $\{w(\mathfrak{C}_1), w(\mathfrak{C}_2)\}$ a single block, again by [7, Lemma 4.9]. □

Lemma 9.1.4 *Let $\mathfrak{S}$ be a monomial in $\mathfrak{N}$ and x an element of $I(d,n)$. For x to dominate $\mathfrak{S}$ it is necessary and sufficient that for every $\alpha = (r, c)$ in $\mathfrak{S}$ there exist $\beta = (R, C)$ in $\mathfrak{S}_x$ with $C \leq c$, $r \leq R$, and $\operatorname{depth}_{\mathfrak{S}_x}\beta \geq \operatorname{depth}_{\mathfrak{S}}\alpha$. (Here $\mathfrak{S}_x$ denotes the distinguished monomial in $\mathfrak{N}$ associated to x as in [7, Proposition 4.3].)*

Proof: The lemma is a corollary of [7, Lemma 4.5] as we now show.

First suppose that x dominates $\mathfrak{S}$. Let $\alpha = (r, c)$ be an element of $\mathfrak{S}$, and C a v-chain in $\mathfrak{S}$ with tail α and length $\operatorname{depth}_{\mathfrak{S}}\alpha$. Since x dominates C, there exists, by [7, Lemma 4.5], a chain in D in $\mathfrak{S}_x$ of length $\operatorname{depth}_{\mathfrak{S}}\alpha$ and tail $\beta = (R, C)$ with $C \leq c$ and $r \leq R$, and we are done with the proof of the necessity.

To prove the sufficiency, let $C : \alpha_1 = (r_1, c_1) > \ldots > \alpha_k = (r_k, c_k)$ be a v-chain in $\mathfrak{S}$. By hypothesis, there exist $\beta_1 = (R_1, C_1), \ldots, \beta_k = (R_k, C_k)$ in $\mathfrak{S}_x$ with $C_i \leq c_i$, $r_i \leq R_i$, and $\operatorname{depth}_{\mathfrak{S}_x}\beta_i = i$ for $1 \leq i \leq k$ (observe that replacing the $\geq$ in the latter condition of the statement by an equality yields an equivalent statement). We claim that $\beta_1 > \ldots > \beta_k$. By [7, Lemma 4.5], it suffices to prove the claim.

Since β_k has depth k in $\mathfrak{S}_x$, there exists a $\beta'_{k-1} = (R'_{k-1}, C'_{k-1})$ of depth $k-1$ in $\mathfrak{S}_x$ such that $\beta'_{k-1} > \beta_k$. It follows from the distinguishedness of $\mathfrak{S}_x$ that that $\beta'_{k-1} = \beta_{k-1}$: if not, then we have two distinct elements of the same depth (namely $k-1$) in $\mathfrak{S}_x$ both dominating α_k, a contradiction. So $\beta_{k-1} > \beta_k$, and the claim is proved by continuing in a similar fashion. □

Let x be an element of $I(d,n)$. Let $\mathfrak{S}_x$ denote the distinguished monomial in $\mathfrak{N}$ associated to x as in [7, Proposition 4.3]. For k a positive integer, let x_k

denote the element of $I(d,n)$ corresponding to the distinguished subset $(\mathfrak{S}_x)_k$. For a monomial $\mathfrak{S}$ of $\mathfrak{N}$, let $\mathfrak{S}_{k,k+1} := \mathfrak{S}_k \cup \mathfrak{S}_{k+1}$. Let $x_{k,k+1}$ denote the element of $I(d,n)$ corresponding to the distinguished monomial $(\mathfrak{S}_x)_{k,k+1}$; let x^k denote the element of $I(d,n)$ corresponding to the distinguished subset $(\mathfrak{S}_x)^k$.

> Caution: For a monomial $\mathfrak{S}$ of $\mathfrak{O}\mathfrak{N}$ and an odd integer j, we have introduced in §7.1 the notation $\mathfrak{S}_{j,j+1}$. That is different from the $\mathfrak{S}_{k,k+1}$ just defined.

Corollary 9.1.5 *x dominates $\mathfrak{S}$ $\Leftrightarrow$ x_k dominates $\mathfrak{S}_k$ $\forall$ k $\Leftrightarrow$ $x_{1,2}$ dominates $\mathfrak{S}_{1,2}$ and x^3 dominates $\mathfrak{S}^3$.*

Proof: The first equivalence is a restatement of the lemma: in the statement of the lemma we could equally well have written $\text{depth}_{\mathfrak{S}_x}\beta = \text{depth}_{\mathfrak{S}}\alpha$. The second follows from the first and the following observations: $(\mathfrak{S}_{1,2})_1 = \mathfrak{S}_1$, $(\mathfrak{S}_{1,2})_2 = \mathfrak{S}_2$, $(\mathfrak{S}^3)_k = \mathfrak{S}_{k+2}$; and $(x_{1,2})_1 = x_1$, $(x_{1,2})_2 = x_2$, $(x^3)_k = x_{k+2}$. $\square$

9.2 Orthogonal analogues of Lemmas of 9.1

Lemma 9.2.2 below is the orthogonal analogue of Lemma 9.1.4 (more precisely, that of the first assertion of Corollary 9.1.5). The following proposition will be used in its proof.

Proposition 9.2.1 *Let x be an element of $I(d)$ and $\mathfrak{S}$ a monomial in $\mathfrak{O}\mathfrak{N}$. Then x $\mathfrak{O}$-dominates $\mathfrak{S}_1^{\text{pr}} \cup \mathfrak{S}_2^{\text{pr}}$ if and only if it $\mathfrak{O}$-dominates every v-chain in $\mathfrak{S}$ of $\mathfrak{O}$-depth at most 2.*

Proof: The "if" part is immediate from definitions (in any case, see also Proposition 7.5.4). For the "only if" part, let C be a v-chain in $\mathfrak{S}$ of $\mathfrak{O}$-depth at most 2. Our goal is to show that x dominates $\mathfrak{S}_C$. For this, it is enough, by Corollary 9.1.5, to show that x_1 dominates $(\mathfrak{S}_C)_1$ and x_2 dominates $(\mathfrak{S}_C)_2$ (by choice of C, $(\mathfrak{S}_C)_k$ is empty for $k \geq 3$).

Let $\alpha' \in (\mathfrak{S}_C)_1$. Choose α in C such that $\alpha' \in \mathfrak{S}_{C,\alpha}$. Choose α_0 in $\mathfrak{S}_1^{\text{pr}}$ such that α_0 dominates α. Since x $\mathfrak{O}$-dominates the singleton v-chain $\{\alpha_0\}$, it follows that x_1 dominates $q_{\{\alpha_0\},\alpha_0}$. We claim that $q_{\{\alpha_0\},\alpha_0}$ dominates α'. To prove the claim, we need only rule out the possibility that α_0 is of type S in $\{\alpha_0\}$ and α of type V in C. Since $\alpha' \in (\mathfrak{S}_C)_1$, it follows from Proposition 5.3.4 (1) that α is the first element of C. In particular, if α is of type V in C, then $p_h(\alpha) \in \mathfrak{N}$, so $p_h(\alpha_0) \in \mathfrak{N}$, and α_0 is of type H in $\{\alpha_0\}$. The claim is thus proved.

Now consider an element of $(\mathfrak{S}_C)_2$. Observe that the length of C is at most 2 (Lemma 6.2.1). So our element is either the horizontal projection $p_h(\alpha)$ of the head α of C, or it is $q_{C,\beta}$ where β is the tail of C. In the first case, let α_0 be as in the previous paragraph, and proceed similarly. It is clear that $p_h(\alpha_0) \in \mathfrak{N}$ (because $p_h(\alpha) \in \mathfrak{N}$); x_2 dominates $p_h(\alpha_0)$ and so also $p_h(\alpha)$.

Now we handle the second case. If $\beta \in \mathfrak{S}_2^{\text{pr}}$, then C is contained in $\mathfrak{S}_1^{\text{pr}} \cup \mathfrak{S}_2^{\text{pr}}$ and there is nothing to prove. So assume that $\mathfrak{O}\text{-depth}_{\mathfrak{S}}(\beta) \geq 3$. Choose a v-chain D in $\mathfrak{S}$ with tail β, $\mathfrak{O}\text{-depth}_D(\beta) \geq 3$, and with the good property

as in Proposition 6.3.3. There occurs in D an element of $\mathfrak{O}$-depth 3, say δ. (Lemma 6.3.1 (5)). Let A denote the part $\delta > \ldots$ of D and C' the part up to but not including δ. There clearly is an element—call it μ—of depth 2 in $\mathfrak{S}_D$ that dominates $q_{D,\beta}$. This element μ belongs to $\mathfrak{S}_{C'}$ (Corollary 6.3.5 (3)). Since D has the good property of Proposition 6.3.3, $C' \subseteq \mathfrak{S}_1^{\mathrm{pr}} \cup \mathfrak{S}_2^{\mathrm{pr}}$, so μ is dominated by an element in $(\mathfrak{S}_x)_2$. In particular, $q_{D,\beta}$ is dominated by the same element of $(\mathfrak{S}_x)_2$.

We are still not done, for it is possible that $q_{D,\beta}$ be β and $q_{C,\beta}$ be $p_v(\beta)$. Suppose that this is the case. Then $\alpha > \beta$ is connected. So $p_h(\alpha) \in \mathfrak{N}$ and the legs of α and β intertwine. As seen above in the third paragraph of the present proof, there is an element of $(\mathfrak{S}_x)_2$ that dominates $p_h(\alpha)$. By the distinguishedness of $\mathfrak{S}_x$, it follows that the element in $(\mathfrak{S}_x)_2$ dominating β is the same as the one dominating $p_h(\alpha)$. By the symmetry of $\mathfrak{S}_x$, this element lies on the diagonal and so dominates $p_v(\beta)$, and, finally, we are done with the proof in the second case. □

Lemma 9.2.2 *Let $\mathfrak{S}$ be a monomial in $\mathfrak{O}\mathfrak{N}$ and x an element of $I(d)$. For x to $\mathfrak{O}$-dominate $\mathfrak{S}$ it is necessary and sufficient that, for every odd integer j, every v-chain in $\mathfrak{S}_j^{\mathrm{pr}} \cup \mathfrak{S}_{j+1}^{\mathrm{pr}}$ is $\mathfrak{O}$-dominated by $x_{j,j+1}$.*

Proof: First suppose that x dominates $\mathfrak{S}$. Let j be an odd integer and let A a v-chain in $\mathfrak{S}_j^{\mathrm{pr}} \cup \mathfrak{S}_{j+1}^{\mathrm{pr}}$. We need to show that $x_{j,j+1}$ dominates $\mathfrak{S}_A$. For this, we may assume that A is maximal (by Corollary 6.1.2). By Corollary 6.1.3 (3), the length of A is at most 2. By Lemma 6.3.1 (5) (b), for every β in $\mathfrak{S}_{j+1}^{\mathrm{pr}}$ there exists α in $\mathfrak{S}_j^{\mathrm{pr}}$ with $\alpha > \beta$. Thus we may assume that the head α of A belongs to $\mathfrak{S}_j^{\mathrm{pr}}$.

It is enough to show (see [7, Lemma 4.5]) that for any v-chain E in $\mathfrak{S}_A$

- the length of E is at most 2;
- there exists an x-dominated monomial in $\mathfrak{N}$ containing E and the head of E is an element of depth at least j in that monomial.

The first of these conditions holds by Proposition 7.5.4. We now show that the second holds.

We may assume that E is maximal in $\mathfrak{S}_A$. By Proposition 5.3.4 (1), the head of E is $q_{A,\alpha}$. Let C a v-chain in $\mathfrak{S}$ with tail α such that $\mathfrak{O}\text{-depth}_C(\alpha) = j$. Let D be the concatenation of C with A. We claim that the monomial $\mathfrak{S}_D$ has the desired properties. That $\mathfrak{S}_D$ is x-dominated is clear (since x $\mathfrak{O}$-dominates $\mathfrak{S}$). By Corollary 6.3.5, it follows that $q_{D,\alpha} = q_{A,\alpha}$ and $\mathfrak{S}_A \subseteq \mathfrak{S}_D$ (in particular that $E \subseteq \mathfrak{S}_D$). By Proposition 6.1.1 (2), $\mathfrak{O}\text{-depth}_D(\alpha) = \mathfrak{O}\text{-depth}_C(\alpha) = j$, that is, $\operatorname{depth}_{\mathfrak{S}_D} q_{D,\alpha} = j$. The proof of the necessity is thus complete.

To prove the sufficiency, proceed by induction on the largest odd integer J such that $\mathfrak{S}_J^{\mathrm{pr}} \cup \mathfrak{S}_{J+1}^{\mathrm{pr}}$ is non-empty. When $J = 1$, there is nothing to prove, for $\mathfrak{S}_1^{\mathrm{pr}} \cup \mathfrak{S}_2^{\mathrm{pr}} = \mathfrak{S}$ and $x_{1,2}$ $\mathfrak{O}$-dominates $\mathfrak{S}_1^{\mathrm{pr}} \cup \mathfrak{S}_2^{\mathrm{pr}}$. So suppose that $J \geq 3$. We implicitly use Corollary 6.3.7 in what follows. By induction, x^3 $\mathfrak{O}$-dominates $\mathfrak{S}^{3,4}$.

Let D be a v-chain in $\mathfrak{S}$. Our goal is to show that x dominates $\mathfrak{S}_D$. Let α be the element of D with $\mathfrak{O}$-depth$_D(\alpha) = 3$—such an element exists, by Lemma 6.3.1 (5) (if there exists in D an element of $\mathfrak{O}$-depth in D exceeding 2); the following proof works also in the case when α does not exist. Let A be the part $\alpha > \ldots$ of D, and C' the part up to but not including α. By Proposition 6.1.1 (2), the $\mathfrak{O}$-depth (in C') of elements of C' is at most 2. By Proposition 9.2.1, $x_{1,2}$ dominates $\mathfrak{S}_{C'}$. By Corollary 6.3.5 (3), $(\mathfrak{S}_D)_{1,2} = \mathfrak{S}_{C'}$ and $(\mathfrak{S}_D)^3 = \mathfrak{S}_A$. Since $A \subseteq \mathfrak{S}^{3,4}$, it follows that x^3 dominates $\mathfrak{S}_A$ (induction hypothesis). Finally, by an application of Corollary 9.1.5, we conclude that x dominates $\mathfrak{S}_D$. □

Corollary 9.2.3 *Let $\mathfrak{S}$ be a a monomial in $\mathfrak{O}\mathfrak{N}$ and x an element of $I(d)$. For x to $\mathfrak{O}$-dominate $\mathfrak{S}$ it is necessary and sufficient that $x_{1,2}$ $\mathfrak{O}$-dominate $\mathfrak{S}_1^{\mathrm{pr}} \cup \mathfrak{S}_2^{\mathrm{pr}}$ and x^3 $\mathfrak{O}$-dominate $\mathfrak{S}^{3,4}$.*

Proof: It is easy to see that $(x^3)_{j,j+1} = x_{j+2,j+3}$; it follows from Proposition 6.3.6 that $(\mathfrak{S}^{3,4})_j^{\mathrm{pr}} \cup (\mathfrak{S}^{3,4})_{j+1}^{\mathrm{pr}} = \mathfrak{S}_{j+2}^{\mathrm{pr}} \cup \mathfrak{S}_{j+3}^{\mathrm{pr}}$. The assertion follows from the lemma. □

9.3 Orthogonal analogues of some lemmas in [7]

The proofs of Propositions 4.1 and 4.2 of [7] are based on assertion 4.9–4.16 (of that paper). Assertion 4.9 being a statement about a single $\mathfrak{S}_k$, it is applicable in the present situation. Since references to it are frequent, we recall it below as Lemma 9.3.1. As to assertions 4.10–4.16 of [7], assertions 9.3.2, 9.3.4–9.3.9 below are their respective analogues.

A *block* of a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}$ means a block of $\mathfrak{S}_{j,j+1}$ in the sense of [7] for some odd integer j.

> Caution: Considering $\mathfrak{S}$ as a monomial in $\mathfrak{N}$, there is the notion of a "block" of $\mathfrak{S}$ as in [7], which has in fact been used in §9.1, and which is different from the notion just defined. Both notions are used and it will be clear from the context which is meant.

Throughout this section $\mathfrak{S}$ denotes a monomial in $\mathfrak{O}\mathfrak{N}$ and j an integer (not necessarily odd).

Lemma 9.3.1 *If $\mathfrak{B}_1, \ldots, \mathfrak{B}_l$ are the blocks in order from left to right of some $\mathfrak{S}_k$, and $w(\mathfrak{B}_1) = (R_1, C_1)$, $w(\mathfrak{B}_2) = (R_2, C_2)$, ..., $w(\mathfrak{B}_l) = (R_l, C_l)$, then*

$$C_1 < R_1 < C_2 < R_2 < \ldots < R_{l-1} < C_l < R_l$$

Proof: This is merely a recall Lemma 4.9 of [7]. In any case it follows easily from the definitions. □

Lemma 9.3.2 *No two elements of $\mathfrak{S}_k(\text{ext}) \cup \mathfrak{S}'_k$ are comparable. More precisely, it is not possible to have elements $\alpha > \beta$ both belonging to $\mathfrak{S}_k(\text{ext}) \cup \mathfrak{S}'_k$.*

PROOF: It follows from Lemma 9.3.1 that $\mathfrak{S}_k \cup \mathfrak{S}'_k$ contains no comparable elements. If k is even, then $\mathfrak{S}_k(\text{ext}) = \mathfrak{S}_k$ (Corollary 7.3.4 (2)); if k is odd, we may assume $\mathfrak{S}_k(\text{ext}) = \mathfrak{S}_k$ (as sets) by increasing the multiplicity of σ_k in $\mathfrak{S}_k^{\text{pr}}$.□

Lemma 9.3.3 *For integers $i \leq k$, there cannot exist $\gamma \in \mathfrak{S}'_i(\text{up})$ and $\beta \in \mathfrak{S}_k^{\text{pr}}$ such that $\beta > \gamma$. For integers $i < k$, there cannot exist $\gamma \in \mathfrak{S}'_i(\text{up})$ and $\beta \in \mathfrak{S}_k^{\text{pr}}$ such that β dominates γ.*

PROOF: Let $\gamma \in \mathfrak{S}'_i(\text{up})$ and $\beta \in \mathfrak{S}_k^{\text{pr}}$. If $i = k$ and $\beta > \gamma$, then we get a contradiction immediately to Lemma 9.3.2. Now suppose that $i < k$ and that β dominates γ. Apply Corollary 6.3.4 (the notation of the corollary being suggestive of how exactly to apply it). Let α be as in its conclusion. The chain $\alpha > \gamma$ contradicts Lemma 9.3.2 in case i is odd and either Lemma 9.3.2 or Proposition 7.5.4 in case i is even. □

Lemma 9.3.4 *For (r, c) in $\mathfrak{S}'$, there exists a unique block $\mathfrak{B}$ of $\mathfrak{S}$ with (r, c) in $\mathfrak{B}'$.*

PROOF: The existence is clear from the definition of $\mathfrak{S}'$. For the uniqueness, suppose that $\mathfrak{B}$ and $\mathfrak{C}$ are two distinct blocks of $\mathfrak{S}$ with (r, c) in both $\mathfrak{B}'$ and $\mathfrak{C}'$. We will show that this leads to a contradiction.

Let i and k be such that $\mathfrak{B} \subseteq \mathfrak{S}_i$ and $\mathfrak{C} \subseteq \mathfrak{S}_k$. From Lemma 4.11 of [7] (of which the present lemma is the orthogonal analogue) it follows that $i \neq k$, so we can assume without loss of generality that $i < k$. By applying the involution # if necessary, we may assume that $(r, c) \in \mathfrak{S}'_i(\text{up})$. Now there exists an element (r, a) in $\mathfrak{C}$ with $a \leq c$ (this follows from the definition of $\mathfrak{C}'$). Clearly $(r, a) \in \mathfrak{S}_k^{\text{pr}}$. Taking $\beta = (r, a)$ and $\gamma = (r, c)$, we get a contradiction to Lemma 9.3.3. □

Lemma 9.3.5 *Let $i < j$ be positive integers.*

1. *Given a block $\mathfrak{B}$ of $\mathfrak{S}_j$, there exists a unique block $\mathfrak{C}$ of $\mathfrak{S}_i$ such that $w(\mathfrak{C}) > w(\mathfrak{B})$.*
2. *Given an element β in $\mathfrak{S}_j(\text{ext}) \cup \mathfrak{S}'_j$, there exists α in $\mathfrak{S}_i$ such that $\alpha > \beta$.*

PROOF: (1): The assertion follows by applying Lemma 9.1.2 with $\mathfrak{B} = \mathfrak{B}$ and $\mathfrak{U} = \mathfrak{S}_i$. We need to make sure however that the lemma can be applied. More precisely, we need to check that for every $\beta = (r, c)$ in $\mathfrak{B}$ there exist $\gamma^1(\beta) = (R^1, C^1)$ and $\gamma^2(\beta) = (R^2, C^2)$ in $\mathfrak{S}_i$ such that $C^1 < c$, $C^2 < R^1$, and $r < R^2$. We may assume $\beta = \beta(\text{up})$, for, if $\beta = \beta(\text{down})$, then $\beta(\text{up})$ also belongs to $\mathfrak{S}_j$ because $\mathfrak{S}_j$ is symmetric, and we can set $\gamma^1(\beta) = \gamma^2(\beta(\text{up}))(\text{down})$, and $\gamma^2(\beta) = \gamma^1(\beta(\text{up}))(\text{down})$—note that these two belong to $\mathfrak{S}_i$ since $\mathfrak{S}_i$ is symmetric.

We consider three cases:

1. β belongs to $\mathfrak{S}$.

2. $\beta = p_h(\sigma_{j-1})$ (in particular, j is even and $\mathfrak{S}$ is truly orthogonal at $j-1$).

3. $\beta = p_v(\sigma_j)$ (in particular, j is odd and $\mathfrak{S}$ is truly orthogonal at j).

Define β' to be β in case 1, σ_{j-1} in case 2, and σ_j in case 3. Let C be a v-chain in $\mathfrak{S}$ with tail β' and having the good property as in Proposition 6.3.3.

First suppose that there exists in C an element of $\mathfrak{O}$-depth i and denote it by γ. If $p_h(\gamma) \notin \mathfrak{N}$ (this can happen only in case 1), then set $\gamma^1(\beta) = \gamma^2(\beta) = \gamma$. Now suppose $p_h(\gamma) \in \mathfrak{N}$. Then $\gamma \in \mathfrak{S}_i$ except when $\gamma = \sigma_i$ with i odd and σ_i has multiplicity 1 in $\mathfrak{S}$. If $\gamma \in \mathfrak{S}_i$, take $\gamma^1(\beta) = \gamma$ and $\gamma^2(\beta) = \gamma^{\#} = \gamma(\text{down})$; if $\gamma \notin \mathfrak{S}_i$, then take $\gamma^1(\beta) = \gamma^2(\beta) = p_v(\gamma)$.

Now suppose that C has no element of $\mathfrak{O}$-depth i. Then, by Lemma 6.3.1 (5), i is even and there exists in C an element of $\mathfrak{O}$-depth $i-1$. This element of C is of type H by Lemma 6.3.1 (1), so $\mathfrak{S}$ is truly orthogonal at $i-1$. Set $\gamma^1(\beta) = \gamma^2(\beta) = p_h(\sigma_{i-1})$.

(2): This proof parallels the proof of (1) above. As in the above proof, we may assume that $\beta = \beta(\text{up})$. Suppose $\beta = (r, c)$ belongs to $\mathfrak{S}'_j$. Then there exists $(r, a) \in \mathfrak{S}_j$ with $a \leq c$. Since $\mathfrak{S}'_j$ does not meet the diagonal, it is clear that $(r, a) \in \mathfrak{O}\mathfrak{N}$, and thus it is enough to prove the assertion for $\beta \in \mathfrak{S}_j(\text{ext})$.

So now take $\beta \in \mathfrak{S}_j(\text{ext})$. Let β' and C be in the proof of (1). First suppose that there exists in C an element of $\mathfrak{O}$-depth i. Denote it by γ. If $\gamma \in \mathfrak{S}_i$, then take $\alpha = \gamma$. If $\gamma \notin \mathfrak{S}_i$, then $p_v(\gamma) \in \mathfrak{S}_i$, and we take $\alpha = p_v(\gamma)$. In case there is no element in C of $\mathfrak{O}$-depth i, we take $\alpha = p_h(\sigma_{i-1})$ (see the above proof). $\square$

Corollary 9.3.6 *If $\mathfrak{B}$ and $\mathfrak{B}_1$ are blocks of $\mathfrak{S}$ with $w(\mathfrak{B}) = (r, c)$ and $w(\mathfrak{B}_1) = (r_1, c_1)$, then exactly one of the following holds:*

$$c < r < c_1 < r_1, \qquad c_1 < r_1 < c < r,$$
$$c < c_1 < r_1 < r, \quad \text{or} \quad c_1 < c < r < r_1.$$

Proof: This is a formal consequence of Lemmas 9.3.1 and 9.3.5, just as Corollary 4.13 of [7] is of Lemmas 4.9 and 4.12 of that paper. $\square$

Corollary 9.3.7 *If $w(\mathfrak{B}) > w(\mathfrak{C})$ for blocks $\mathfrak{B} \subseteq \mathfrak{S}_i$ and $\mathfrak{C} \subseteq \mathfrak{S}_j$ of $\mathfrak{S}$, then $i < j$.*

Proof: This is a formal consequence of Lemmas 9.3.1 and 9.3.5. It follows from the first lemma that $i \neq j$. Suppose $i > j$. Then there exists by the second lemma a block $\mathfrak{C}' \subseteq \mathfrak{S}_j$ such that $w(\mathfrak{C}') > w(\mathfrak{B})$. But then $w(\mathfrak{C}') > w(\mathfrak{C})$, a contradiction of the first lemma. $\square$

Corollary 9.3.8 *Let $(s, t) > (s_1, t_1)$ be elements of $\mathfrak{S}'$, and $\mathfrak{B}$, $\mathfrak{B}_1$ be blocks of $\mathfrak{S}$ such that $(s, t) \in \mathfrak{B}'$, and $(s_1, t_1) \in \mathfrak{B}'_1$. Then $w(\mathfrak{B}) > w(\mathfrak{B}_1)$.*

PROOF: Let $w(\mathfrak{B}) = (r, c)$ and $w(\mathfrak{B}_1) = (r_1, c_1)$. By Corollary 9.3.6, we have four possibilities. Since (r, c) dominates (s, t) and (r_1, c_1) dominates (s_1, t_1), the possibilities $c < r < c_1 < r_1$ and $c_1 < r_1 < c < r$ are eliminated. It is thus enough to eliminate the possibility $c_1 < c < r < r_1$. Suppose that this is the case. Then, by Corollary 9.3.7, $j_1 < j$, where j_1 and j are such that $\mathfrak{B} \subseteq \mathfrak{S}_j$ and $\mathfrak{B}_1 \subseteq \mathfrak{S}_{j_1}$. Now, by Lemma 9.3.5 (2), there exists α in $\mathfrak{S}_{j_1}$ such that $\alpha > (s, t) > (s_1, t_1)$. But then this contradicts Lemma 9.3.2. $\square$

Corollary 9.3.9 *For a $\mathfrak{B} \subseteq \mathfrak{S}_i$ of $\mathfrak{S}$, the depth of $w(\mathfrak{B})$ in $\mathfrak{S}_w$ is exactly i.*

PROOF: That the depth is at least i follows from Lemma 9.3.5. That the depth cannot exceed i follows from Corollary 9.3.7. $\square$

Corollary 9.3.10 *Let $\alpha \in \mathfrak{S}'_k(\mathrm{up})$, $\beta \in \mathfrak{S}'_m(\mathrm{up})$, and $\alpha > \beta$. Then $k < m$.*

PROOF: Corollary 9.3.8 and Corollary 9.3.9. $\square$

9.4 More lemmas

This subsection is a collection of lemmas to be invoked in the later subsections. More specifically, Lemma 9.4.1 and Corollary 9.4.2 are invoked in the proof of Proposition 4.1.1 in §10.1, Lemma 9.4.3 in the proof of the first half of Proposition 4.1.2 in §10.2, and Lemma 9.4.4 in the proof of the second half of Proposition 4.1.2 in §10.3. Throughout this subsection, $\mathfrak{S}$ denotes a monomial in $\mathfrak{O}\mathfrak{N}$.

Lemma 9.4.1 *Let C be a v-chain in $\mathfrak{S}'$, α an element of C, and $\alpha' \in \mathfrak{S}_{C,\alpha}$. Then $\mathrm{depth}_{\mathfrak{S}_C} \alpha' \leq k(\mathrm{even})$, for k the integer such that $\alpha \in \mathfrak{S}'_k(\mathrm{up})$.*

PROOF: Proceed by induction on k. If $k = 1$, the assertion follows from Corollary 9.3.10, so assume $k > 1$. Choose a v-chain C' in $\mathfrak{S}_C$ with tail α' and $\mathrm{depth}_{C'} \alpha' = \mathrm{depth}_{\mathfrak{S}_C}(\alpha')$. The length of a v-chain in $\mathfrak{S}_{C,\alpha}$ is clearly at most 2. So, if γ' is the element two steps before α' in C' (if γ' does not exist then there is clearly nothing to prove), then $\gamma' \in \mathfrak{S}_{C,\gamma}$ with $\gamma > \alpha$ (see Proposition 5.3.4 (2)). We claim that $\mathrm{depth}_{\mathfrak{S}_C}(\gamma') \leq k(\mathrm{odd}) - 1$. It is enough to prove the claim, for then $\mathrm{depth}_{\mathfrak{S}_C}(\alpha') = \mathrm{depth}_{C'} \alpha' = \mathrm{depth}_{C'} \gamma' + 2 \leq k(\mathrm{odd}) - 1 + 2 = k(\mathrm{even})$.

The claim follows by induction from Corollary 9.3.10 if k is odd or more generally if $\gamma \in \mathfrak{S}'_l(\mathrm{up})$ with $l \leq k(\mathrm{odd}) - 1$. So assume that k is even and $\gamma \in \mathfrak{S}'_{k-1}(\mathrm{up})$. By 7.5.4, it is not possible that γ is of type H and $p_h(\gamma) > \alpha$. So the only possibility is that $\alpha' = p_h(\alpha)$ and $\gamma > \alpha$ is connected. In particular, γ is of type V and α of type H in C and $\gamma' = p_v(\gamma)$.

Now let μ be the first element in the connected component of α in C. The cardinality of the part $\mu > \ldots > \gamma$ of C is even (by Proposition 5.3.1 (1), it follows that the cardinality of $\mu > \ldots > \alpha$ is odd), say e. Letting m be such that $\mu \in \mathfrak{S}'_m(\mathrm{up})$, we have, by Proposition 9.3.10, $m \leq k - 1 - (e - 1) =$

$k - e$. If $m(\text{even}) < k - e$, then, since $\text{depth}_{C'}\gamma' = \text{depth}_{C'}p_v(\mu) + e - 1$ (by Proposition 5.3.4 (1), since, by Proposition 5.3.1 (2), $\mu, \ldots, \gamma$ all have type V in C) and $\text{depth}_{C'}p_v(\mu) \leq m(\text{even})$ by induction, it follows that $\text{depth}_{C'}\gamma' < k - e + e - 1 = k - 1$, and we are done.

So suppose that $m(\text{even}) = k - e$. Let ν be the element just before μ in C (if such an element does not exist, then $\text{depth}_{C'}\gamma' = e \leq k - 2$—observe that $m(\text{even}) \geq 2$—and we are done). Then $\nu > \mu$ is not connected (by choice of μ). So $p_h(\nu) > \mu$. By Proposition 7.5.4, this means that $j \leq m(\text{even}) - 2$ where j is the odd integer defined by $\nu \in \mathfrak{S}'_j(\text{up}) \cup \mathfrak{S}'_{j+1}(\text{up})$. So, again by induction, $\text{depth}_{C'}\gamma' = \text{depth}_{C'}p_h(\nu) + e \leq m(\text{even}) - 2 + e = k - 2$, and the claim is proved. $\square$

Corollary 9.4.2 *The $\mathfrak{O}$-depth of an element α in $\mathfrak{S}'$ is at most k where k is such that $\alpha \in \mathfrak{S}'_k(\text{up})$.*

Proof: Let C' be a v-chain in $\mathfrak{S}_C$ with tail $q_{C,\alpha}$. If k is even, then, by the lemma, $\text{depth}_{C'}q_{C,\alpha} \leq k$. So suppose that k is odd. Let γ' be the immediate predecessor of $q_{C,\alpha}$ in C'. By Proposition 5.3.4 (2), $\gamma > \alpha$, and so $\gamma \in \mathfrak{S}'_l(\text{up})$ with $l \leq k - 1$ (see the observation in the first paragraph of the proof of the lemma). So $\text{depth}_{C'}\gamma' \leq k - 1$ (by the lemma) and $\text{depth}_{C'}\alpha' = \text{depth}_{C'}\gamma' + 1 \leq k$. $\square$

Lemma 9.4.3 *Let $\mathfrak{S}$ be a monomial in $\mathfrak{O}\mathfrak{N}$ and $\mathfrak{O}\pi(\mathfrak{S}) = (w, \mathfrak{S}')$. Let $i < k$ be integers, α an element of $\mathfrak{S}'_i(\text{up})$, and δ an element of $(\mathfrak{S}_w)_k(\text{up})$ that dominates α.*

1. *If k is even, then there exists $\beta \in \mathfrak{S}'_k(\text{up})$ with $\alpha > \beta$.*
2. *If k is odd and $w_{k,k+1}$ $\mathfrak{O}$-dominates the singleton v-chain α, then either there exists $\beta \in \mathfrak{S}'_k(\text{up})$ with $\alpha > \beta$ or there exists $\gamma \in \mathfrak{S}'_{k+1}(\text{up})$ with $p_h(\alpha) > \gamma$.*

Proof: Write $\alpha = (r, c)$ and $\delta = (A, B)$. By Corollary 9.3.9, there exists a block $\mathfrak{B}$ of $\mathfrak{S}_k$ such that $\delta = w(\mathfrak{B})$. Let (D, B) be the first element of $\mathfrak{B}$ (arranged in increasing order of row and column indices). We have the following possibilities:

(i) $D \leq A$ and $(D, B) \in \mathfrak{S}^{\text{pr}}_k$.

(ii) k is odd, $\mathfrak{S}$ is truly orthogonal at k, $(D, B) = (A, B) = p_v(\sigma_k)$, and $\mathfrak{B}$ consists of the single diagonal element $(D, B) = (B^*, B)$.

(iii) k is even, $\mathfrak{S}$ is truly orthogonal at $k-1$, $(D, B) = (A, B) = p_h(\sigma_{k-1})$, and $\mathfrak{B}$ consists of the single diagonal element $(D, B) = (B^*, B)$.

We claim the following: in case (i), $D < r$ (in particular, $D < A$); in case (ii), the row index of σ_k is less than r; and case (iii) is not possible. The first two assertions and also the third in the case $i < k - 1$ follow readily from

Lemma 9.3.3; in case (iii) holds and $i = k-1$, then $\sigma_{k-1} > \alpha$, a contradiction to Lemma 9.3.2.

First suppose that possibility (ii) holds. Write $\sigma_k = (s, B)$. Since $s < r$ and $p_h(\sigma_k) \in \mathfrak{N}$, it is clear that $p_h(\alpha) = (r, r^*)$ also belongs to $\mathfrak{N}$. From the hypothesis that $w_{k,k+1}$ $\mathfrak{O}$-dominates $\{\alpha\}$, it follows that there is an element of $(\mathfrak{S}_w)_{k+1}$ that dominates $p_h(\alpha) = (r, r^*)$. Such an element must be diagonal (because of the distinguishedness of $\mathfrak{S}_w$), and so must be the $w(\mathfrak{C})$ for the unique diagonal block $\mathfrak{C}$ of $\mathfrak{S}_{k+1}$. In particular, this means that there are elements other than (s, s^*) in $\mathfrak{S}_{k+1}$, and so $\mathfrak{S}'_{k+1}$ is non-empty. In the arrangement of elements of $\mathfrak{S}'_{k+1}(\text{up})$ in increasing order of row and column numbers, let $\gamma = (e, s^*)$ be the last element. Then $e < s < r$ and $r^* < s^*$, so $p_h(\alpha) > \gamma$, and we are done.

Now suppose that possibility (i) holds. Let (p, q) be the element of $\mathfrak{S}_k$ such that p is the largest row index that is less than r, and, among those elements with row index p, the maximum possible column index is q. The arrangement of elements of $\mathfrak{S}_k$ (in increasing order or row and column indices) looks like this:

$$\ldots, (p, q), (s, t), \ldots$$

Since $p < r \leq A$ and $w(\mathfrak{B}) = (A, B)$, we can be sure that (p, q) is not the last element of $\mathfrak{B}$.

We first consider the case $c < t$. Then $\alpha = (r, c) > \beta := (p, t) \in \mathfrak{S}'_k$. If $\beta \in \mathfrak{S}'_k(\text{up})$, then we are done. It is possible that (p, q) lies on or below the diagonal so that β lies below the diagonal, in which case, $\alpha > \beta(\text{up})$ and $\beta(\text{up}) \in \mathfrak{S}'_k(\text{up})$, and again we are done.

Now suppose that $t \leq c$. We claim that:

- (s, t) belongs to the diagonal;
- k is odd and $\mathfrak{S}$ is truly orthogonal at k; and
- $\sigma_k = (u, t)$ with $u < r$.

Suppose that (s, t) does not belong to the diagonal. Since $r \leq s$ (by choice of (p, q)), it follows that (s, t) dominates (r, c). This leads to a contradiction to Lemma 9.3.3, for either (s, t) or its reflection (t^*, s^*) (whichever is above the diagonal) belongs to $\mathfrak{S}^{\text{pr}}_k$ and dominates $\alpha = (r, c)$ in $\mathfrak{S}'_i(\text{up})$. This shows that (s, t) belongs to the diagonal. If k is even, then $(s, t) = p_h(\sigma_{k-1})$, which means $\sigma_{k-1} > \alpha$, again contradicting Lemma 9.3.3, so k must be odd. It also follows that $\mathfrak{S}$ is truly orthogonal at k and that $(s, t) = p_v(\sigma_k)$. Writing $\sigma_k = (u, t)$, if $r \leq u$, then σ_k would dominate α, again contradicting Lemma 9.3.3. So $u < r$, and the claim is proved.

To finish the proof of the lemma, now proceed as in the proof when possibility (ii) holds. $\square$

Lemma 9.4.4 *Let $\mathfrak{T}$ be a monomial in $\mathfrak{O}\mathfrak{N}$ and w an element of $I(d)$ that $\mathfrak{O}$-dominates $\mathfrak{T}$. Let $\beta' > \beta$ be elements $\mathfrak{S}_w(\text{up})$. Let $d-1$ and d be their respective depths in $\mathfrak{S}_w$. Let α be an element of $\mathfrak{O}\mathfrak{P}^*_\beta$ or more generally an element of $\mathfrak{O}\mathfrak{N}$ such that*

(a) it is dominated by β,

(b) it is not comparable to any element of $\mathfrak{P}_\beta$, and

(c) in case d is odd, then $\{\alpha\} \cup \mathfrak{T}_{w,d,d+1}$ has $\mathfrak{O}$-depth at most 2.

Then

1. *there exists $\alpha' \in \mathfrak{P}^*_{\beta'}(\mathrm{up})$ with $\alpha' > \alpha$;*
2. *for α' as in (1), if α' is diagonal, then $p_h(\delta_{d-2}) > \alpha$ if d is odd and $\delta_{d-1} > \alpha$ if d is even.*

PROOF: Assertion (2) is rather easy to prove. If d is odd, then, in fact, $p_h(\delta_{d-2}) = \alpha'$; if d is even, then δ_{d-1} has the same column index as α' and, by Proposition 8.2.2 (4), has row index more than that of α, so $\delta_{d-1} > \alpha$.

Let us prove (1). Write $\alpha = (r, c)$, $\beta = (R, C)$, and $\beta' = (R', C')$. There exists, by the definition of $\mathfrak{P}^*_{\beta'}$, an element in $\mathfrak{P}^*_{\beta'}$ with column index C'. We have $C' < c$ (for $C' < C \leq c$). Let (r', c') be the element of $\mathfrak{P}^*_{\beta'}$ such that c' is maximum possible subject to $c' < c$ and among those elements with column index c' the maximum possible row index is r'. If $r < r'$, then we are done (if (r', c') is below the diagonal, its mirror image would have the desired properties). It suffices therefore to suppose that $r' \leq r$ and arrive at a contradiction.

In the arrangement of elements of $\mathfrak{P}^*_{\beta'}$ in non-decreasing order of row and column indices, there is a portion that looks like this:

$$\ldots,\ (r', c'),\ (a, b),\ \ldots$$

Since there is in $\mathfrak{P}^*_\beta$ an element with row index R' (and clearly $r' \leq r < R < R'$), it follows that (a, b) exists (that is, (r', c') is not the last element in the above arrangement). It follows from the construction of $\mathfrak{P}^*_{\beta'}$ from $\mathfrak{P}_{\beta'}$ that (r', b) is an element in $\mathfrak{P}_{\beta'}$. By the choice of (r', c'), we have $c \leq b$. Thus (r, c) dominates (r', b).

The proof now splits into two cases accordingly as d is even or odd. First suppose that d is even. Then, since β dominates (r', b) and yet (r', b) does not belong to $\mathfrak{P}_\beta$, there exists a v-chain in $\mathfrak{T}_{w,d-1,d}$ of length 2 and head (r', b). The tail of this v-chain then belongs to $\mathfrak{P}_\beta$ and is dominated by (r, c), a contradiction to our assumption that α is not comparable to any element of $\mathfrak{P}_\beta$.

Now suppose that d is odd. Choose a v-chain C in $\mathfrak{T}$ with head (r', b) that is not $\mathfrak{O}$-dominated by w^d. Let D be the part of C consisting of elements of $\mathfrak{O}$-depth (in C) at most 2. We claim that D is $\mathfrak{O}$-dominated by $w_{d,d+1}$. In fact, we claim the following: Any v-chain F with head (r', b) and $\mathfrak{O}$-depth at most 2 is $\mathfrak{O}$-dominated by $w_{d,d+1}$.

To prove the claim, we first prove the following subclaim:

> (†) If the horizontal projection of (r', b) belongs to $\mathfrak{N}$, then β is on the diagonal and dominates the vertical projection of (r', b), and the diagonal element β_1 of $(\mathfrak{S}_w)_{d+1}$ dominates the horizontal projection of (r', b).

Let $p_h(r',b) \in \mathfrak{N}$. Then β belongs to the diagonal because $\mathfrak{S}_w$ is distinguished and symmetric. Once β is on the diagonal, it is clear that it dominates $p_v(r',b)$ (from our assumptions, β dominates (r,c) and (r,c) dominates (r',b)). It follows from Proposition 8.2.2 (3) that the row index of β_1 exceeds the row index r of (r,c), so β_1 dominates $p_h(r',b)$. This finishes the proof of the subclaim (†).

To begin the proof of the claim, observe that F has length at most 2. Suppose first that F consists only of the single element (r',b). The type of (r',b) in F is either H or S. If it is S, then since β dominates (r',b), the claim follows immediately. If it is H, then the claim follows immediately from the subclaim (†).

Continuing with the proof of the claim, let now F consist of two elements: $(r',b) > \mu$. Let γ be the element of $\mathfrak{S}_w$ such that $\mu \in \mathfrak{P}_\gamma$, and let e be the depth of γ in $\mathfrak{S}_w$. From Lemma 8.2.1 (2b) it follows that $e \geq d$. If $e = d$, then $\gamma = \beta$ (by the distinguishedness of $\mathfrak{S}_w$), and the comparability of (r,c) and μ contradicts our hypothesis (b). So $e \geq d+1$, and there exists δ of depth $d+1$ in $\mathfrak{S}_w$ that dominates μ. We have $\beta > \delta$ (again by the distinguishedness of $\mathfrak{S}_w$).

The possibilities for the types of (r',b) and μ in F are: S and S, V and V, H and S (in the last case $p_h(r',b) \not> \mu$ by Lemma 6.3.1 (1)). Noting the existence in $(\mathfrak{S}_w)_{d,d+1}$ of the v-chain $\beta > \delta$ in the first case and also of $\beta > \beta_1$ (where β_1 is as in the subclaim) in the last case, the proof of the claim in these cases is over. So suppose that the second possibility holds. The distinguishedness of $\mathfrak{S}_w$ implies that $\delta = \beta_1$. Since δ is diagonal, it dominates the vertical projection of μ. Noting the existence of the v-chain in $\beta > \delta$ in $(\mathfrak{S}_w)_{d,d+1}$, the proof of the claim in this case too is over.

We continue with the proof of the lemma. It follows from the claim that D is $\mathfrak{O}$-dominated by $w_{d,d+1}$. From Corollary 9.2.3 it follows that the complement E of D in C is not $\mathfrak{O}$-dominated by $w_{d+2,d+3}$ (in particular, that E is non-empty) and that every v-chain in $\mathfrak{T}$ with head ϵ (where ϵ denotes the head of E) is $\mathfrak{O}$-dominated by w^d (given such a v-chain, the concatenation of D with it is $\mathfrak{O}$-dominated by w^{d-2}, and ϵ continues to have $\mathfrak{O}$-depth 3 in the concatenated v-chain). Thus ϵ belongs to $\mathfrak{T}_{w,d,d+1}$. From (1) and (2b) of Lemma 8.2.1 it follows that the element μ of C in between (r',b) and ϵ (if it exists at all) also belongs to $\mathfrak{T}_{w,d,d+1}$. Now consider the v-chain obtained as follows: take the part of C up to (and including) ϵ and replace its head (r',b) by (r,c). This chain has $\mathfrak{O}$-depth 3 and lives in $\{\alpha\} \cup \mathfrak{T}_{w,d,d+1}$, a contradiction to hypothesis (c). □

Corollary 9.4.5 *Let $\mathfrak{T}$ be a monomial in $\mathfrak{O}\mathfrak{N}$ and w an element of $I(d)$ that $\mathfrak{O}$-dominates $\mathfrak{T}$. Let $\beta' > \beta$ be elements of $\mathfrak{S}_w(\mathrm{up})$, α an element of $\mathfrak{O}\mathfrak{P}^*_\beta$, and $d' := \mathrm{depth}_{\mathfrak{S}_w}\beta'$.*

1. *If d' is odd, there exists $\alpha' \in \mathfrak{O}\mathfrak{P}^*_{\beta'}$ such that $\alpha' > \alpha$.*
2. *If there does not exist $\alpha' \in \mathfrak{O}\mathfrak{P}^*_{\beta'}$ such that $\alpha' > \alpha$ then (d' is even by (1) above and) there exists $\alpha'' \in \mathfrak{O}\mathfrak{P}^*_{\beta''}$ such that $p_h(\alpha'') > \alpha$, where β'' is the unique element of $(\mathfrak{S}_w)_{d'-1}$ such that $\beta'' > \beta'$.*

Proof: Immediate from the lemma. □

Corollary 9.4.6 *Let $\mathfrak{T}$ be a monomial in $\mathfrak{OM}$ and w an element of $I(d)$ that $\mathfrak{O}$-dominates $\mathfrak{T}$. Let β, β' be elements of $\mathfrak{S}_w(\mathrm{up})$, and α, α' elements of $\mathfrak{OP}^*_\beta$ and $\mathfrak{OP}^*_{\beta'}$ respectively.*

1. *If $\alpha' > \alpha$ then $\beta' > \beta$ (in particular, $\mathrm{depth}_{\mathfrak{S}_w}\beta' < \mathrm{depth}_{\mathfrak{S}_w}\beta$).*
2. *If $p_h(\alpha') > \alpha$ and $\mathrm{depth}_{\mathfrak{S}_w}\beta$ is even, $\mathrm{depth}_{\mathfrak{S}_w}\beta' \leq \mathrm{depth}_{\mathfrak{S}_w}\beta - 2$.*

Proof: (1) Writing $\beta = (r, c)$ and $\beta' = (r', c')$, there are, since both β and β' dominate α and $\mathfrak{S}_w$ is distinguished, the following four possibilities:

$$c < r < c_1 < r_1, \quad c_1 < r_1 < c < r, \quad c < c_1 < r_1 < r, \quad c_1 < c < r < r_1$$

Since $\alpha' > \alpha$, and α, α' are dominated respectively by β, β' (this is because α, α' belong to $\mathfrak{OP}^*_\beta$, $\mathfrak{OP}^*_{\beta'}$ respectively), the possibilities $c < r < c_1 < r_1$ and $c_1 < r_1 < c < r$ are eliminated (by the distinguishedness of $\mathfrak{S}_w$). It is thus enough to eliminate the possibility $\beta > \beta'$. Suppose, by way of contradiction, that $\beta > \beta'$. By Corollary 9.4.5, either there exists $\gamma \in \mathfrak{OP}^*_\beta$ such that $\gamma > \alpha'$, in which case the v-chain $\gamma > \alpha$ in $\mathfrak{OP}^*_\beta$ contradicts Proposition 8.2.2 (3) or (4), or $d := \mathrm{depth}_{\mathfrak{S}_w}\beta$ is even and there exists (with β'' being the unique element in $\mathfrak{S}_w$ such that $\beta'' > \beta$ and $\mathrm{depth}_{\mathfrak{S}_w}\beta'' = d - 1$) an element $\alpha'' \in \mathfrak{OP}^*_{\beta''}$ with $p_h(\alpha'') > \alpha'$, in which case the v-chain $\alpha'' > \alpha$ in $\mathfrak{T}^\star_{w,d-1,d}$ has $\mathfrak{O}$-depth 3 and so contradicts Proposition 8.2.2 (2).

(2) Set $d := \mathrm{depth}_{\mathfrak{S}_w}\beta$. If $\mathrm{depth}_{\mathfrak{S}_w}\beta'$ were $d-1$, then the v-chain $\alpha' > \alpha$ in $\mathfrak{T}^\star_{w,d-1,d}$ would be of $\mathfrak{O}$-depth 3 and so would contradict Proposition 8.2.2 (2). $\square$

10 The Proof

The aim of this section is to prove Propositions 4.1.1 and 4.1.2. The proof of first proposition appears in §10.1 and that of the second in §§10.2, 10.3. In §9.4 some lemmas are established that are used in the proofs. Needless to say that the lemmas maybe unintelligible until one tries to read the proofs in the later subsections.

10.1 Proof of Proposition 4.1.1

(1) By definition, w is the element of $I(d)$ associated to the distinguished monomial $\cup_k \mathfrak{S}_{w(k)}$. By the very definition of this association, we have $w \geq v$.

(2) This follows from the corresponding property of the map π of [7]. More precisely, that property justifies the third equality below. The other equalities

are clear from the definitions.

$$\begin{aligned}
v\text{-degree}(w) + \text{degree}(\mathfrak{S}') &= \frac{1}{2}\text{degree}(\mathfrak{S}_w) + \frac{1}{2}\sum_k \text{degree}(\mathfrak{S}'_k) \\
&= \frac{1}{2}\sum_k \big(\text{degree}(\mathfrak{S}_{w(k)}) + \text{degree}(\mathfrak{S}'_k)\big) \\
&= \frac{1}{2}\sum_k \text{degree}(\mathfrak{S}_k) \\
&= \frac{1}{2}\sum_{j \text{ odd}} \text{degree}(\mathfrak{S}_{j,j+1}) \\
&= \sum_{j \text{ odd}} \big(\text{degree}(\mathfrak{S}^{\text{pr}}_j) + \text{degree}(\mathfrak{S}^{\text{pr}}_{j+1})\big) \\
&= \text{degree}(\mathfrak{S})
\end{aligned}$$

(3) We have:

$$\begin{aligned}
w\ \mathfrak{O}\text{-dominates } \mathfrak{S}' &\Leftrightarrow w \geq w_C\ \forall\ v\text{-chain } C \text{ in } \mathfrak{S}' \\
&\Leftrightarrow w \text{ dominates } \mathfrak{S}_C\ \forall\ v\text{-chain } C \text{ in } \mathfrak{S}' \\
&\Leftrightarrow \forall\ v\text{-chain } C \text{ in } \mathfrak{S}',\ \forall\ \alpha' = (r,c) \in \mathfrak{S}_C, \\
&\qquad \exists\ \beta = (R,C) \in \mathfrak{S}_w \text{ with } C \leq c,\ r \leq R, \\
&\qquad \text{and depth}_{\mathfrak{S}_w}\beta \geq \text{depth}_{\mathfrak{S}_C}\alpha'.
\end{aligned}$$

The first equivalence above follows from the definition of $\mathfrak{O}$-domination, the second from [7, Lemma 4.5], the third from Lemma 9.1.4.

Now let C be a v-chain in $\mathfrak{S}'$ and $\alpha' = (r,c)$ in $\mathfrak{S}_C$. We will show that there exists β in $\mathfrak{S}_w$ that dominates α and satisfies $\text{depth}_{\mathfrak{S}_w}\beta \geq \text{depth}_{\mathfrak{S}_C}\alpha'$. Let α be the element in C such that $\alpha' \in \mathfrak{S}_{C,\alpha}$, let k be such that $\alpha \in \mathfrak{S}'_k(\text{up})$, and let $\mathfrak{B}$ be the block of $\mathfrak{S}_k$ such that $\alpha \in \mathfrak{B}'$. Writing $\alpha = (r_1, c_1)$ and $w(\mathfrak{B}) = (R_1, C_1)$, we have $C_1 \leq c_1$ and $r_1 \leq R_1$ straight from the definition of $w(\mathfrak{B})$. By Corollary 9.3.9, $\text{depth}_{\mathfrak{S}_w} w(\mathfrak{B}) = k$.

First suppose that $w(\mathfrak{B})$ dominates α' (meaning $C_1 \leq c$ and $r \leq R_1$). If $k \geq \text{depth}_{\mathfrak{S}_C}\alpha'$, we are clearly done; by Corollary 9.4.2, this is the case when $\alpha' = q_{C,\alpha}$. So suppose that α is of type H, $\alpha' = p_h(\alpha)$, and that $k < \text{depth}_{\mathfrak{S}_C}\alpha'$. By Lemma 9.4.1, $\text{depth}_{\mathfrak{S}_C}\alpha' \leq k(\text{even})$. It follows that k is odd and $\text{depth}_{\mathfrak{S}_C}\alpha' = k+1$. By Corollary 7.5.3, $\mathfrak{S}$ is truly orthogonal at k, which means that $\mathfrak{S}_{k+1}$ has a diagonal block, say $\mathfrak{C}$. Note that $w(\mathfrak{C})$ dominates $p_h(\sigma_k)$ which in turn dominates $p_h(\alpha)$. Since $\text{depth}_{\mathfrak{S}_w} w(\mathfrak{C}) = k+1$ by Corollary 9.3.9, we are done.

Now suppose that $w(\mathfrak{B})$ does not dominate α'. Then $\mathfrak{B}$ is non-diagonal and $\alpha' = p_v(\alpha)$. Since $\mathfrak{B}$ is non-diagonal, $p_h(\alpha) \notin \mathfrak{N}$, and α cannot be of type H. So α is of type V in C. It follows easily (see Proposition 5.3.1 (3)) that α is the critical element in C, and and that last element in its connected component in C; by Lemma 6.3.1 (4), $\mathfrak{O}\text{-depth}_C(\alpha) = \text{depth}_{\mathfrak{S}_C} q_{C,\alpha} =: d$ is even. By Proposition 5.3.1 (1), (2), the cardinality of the connected component of α in C is even.

The immediate predecessor γ of α in C is connected to α (this follows from what has been said above). It is of type V in C, $p_h(\gamma)$ belongs to $\mathfrak{N}$, and $\operatorname{depth}_{\mathfrak{S}_C} p_v(\gamma) = d-1$ (see Lemma 6.3.1 (1)). Let ℓ be such that $\gamma \in \mathfrak{S}'_\ell(\mathrm{up})$. Let $\mathfrak{C}$ be the block of $\mathfrak{S}_\ell$ such that $\gamma \in \mathfrak{C}'$. Since $p_h(\gamma) \in \mathfrak{N}$, $\mathfrak{C}$ is diagonal. Note that $w(\mathfrak{C})$ dominates $p_v(\gamma)$ and that $p_v(\gamma) > p_v(\alpha)$. By Corollary 9.3.9, $\operatorname{depth}_{\mathfrak{S}_w} w(\mathfrak{C}) = \ell$. Thus if $d \le \ell$ we are done. On the other hand, $d-1 \le \ell$ by Corollary 9.4.2.

So we may assume that $\ell = d-1$. By Corollary 7.5.3, $\mathfrak{S}$ is truly orthogonal at $d-1$. This implies that $\mathfrak{S}_d$ has a diagonal block, say $\mathfrak{D}$. Note that $w(\mathfrak{D})$ dominates $p_h(\sigma_{d-1})$ which in turn dominates $p_h(\gamma)$. Writing $\gamma = (r_2, c_2)$, since $\gamma > \alpha$ is connected, it follows that (r_1, r_2^*) belongs to $\mathfrak{O}\mathfrak{N}$. Now both $w(\mathfrak{B})$ and $w(\mathfrak{D})$ dominate (r_1, r_2^*). Since $\mathfrak{S}_w$ is distinguished and symmetric and $w(\mathfrak{B})$ is not on the diagonal, it follows that $w(\mathfrak{D}) > w(\mathfrak{B})$. This implies, since $w(\mathfrak{D})$ is on the diagonal, $w(\mathfrak{D}) > p_v(\alpha)$. Since $\operatorname{depth}_{\mathfrak{S}_w} w(\mathfrak{D}) = d$ by Corollary 9.3.9, we are done.

(4) Let x be an element of $I(d)$ that $\mathfrak{O}$-dominates $\mathfrak{S}$. We will show that $x \ge w$. By [7, Lemma 5.5], it is enough to show that x dominates $\mathfrak{S}_w$. By Lemma 9.1.4, it is enough to show the following: for every block $\mathfrak{B}$ of $\mathfrak{S}$, there exists β in $\mathfrak{S}_x$ such that β dominates $w(\mathfrak{B})$ and $\operatorname{depth}_{\mathfrak{S}_x}\beta \ge \operatorname{depth}_{\mathfrak{S}_w} w(\mathfrak{B})$.

Let $\mathfrak{B}$ be a block of $\mathfrak{S}$. By Corollary 9.3.9, $\operatorname{depth}_{\mathfrak{S}_w} w(\mathfrak{B}) = k$ where $\mathfrak{B} \subseteq \mathfrak{S}_k$. Let $\mathfrak{S}_x^k$ denote the set of elements of $\mathfrak{S}_x$ of depth at least k. Our goal is to show that there exists β in $\mathfrak{S}_x^k$ that dominates $w(\mathfrak{B})$. It follows easily from the distinguishedness of $\mathfrak{S}_x$ and the fact that $\mathfrak{B}$ is a block, that it suffices to show the following: given $\alpha \in \mathfrak{B}$, there exists β in $\mathfrak{S}_x^k$ (depending upon α) that dominates α. Moreover, since $\mathfrak{B}$ and $\mathfrak{S}_x^k$ are symmetric, we may assume that $\alpha = \alpha(\mathrm{up})$.

So now let $\alpha = \alpha(\mathrm{up})$ belong to $\mathfrak{B}$. Then either

1. α belongs to $\mathfrak{S}_k^{\mathrm{pr}}$, or
2. k is odd, $\mathfrak{S}$ is truly orthogonal at k, and $\alpha = p_v(\sigma_k)$, or
3. k is even, $\mathfrak{S}$ is truly orthogonal at $k-1$, and $\alpha = p_h(\sigma_{k-1})$.

The proofs in the three cases are similar. In the first case, choose a v-chain C in $\mathfrak{S}$ with tail α such that $\mathfrak{O}\text{-}\operatorname{depth}_C(\alpha) = k$ (see Corollary 6.1.3 (1)). Then $\operatorname{depth}_{\mathfrak{S}_C} q_{C,\alpha} = k$ and, clearly, $q_{C,\alpha}$ dominates α. Since x dominates $\mathfrak{S}_C$, there exists, by Lemma 4.5 of [7], β in $\mathfrak{S}_x^k$ that dominates $q_{C,\alpha}$ (and so also α).

In the second case, choose a v-chain C in $\mathfrak{S}$ with tail σ_k with the property that $\mathfrak{O}\text{-}\operatorname{depth}_C(\sigma_k) = k$. Then $\operatorname{depth}_{\mathfrak{S}_C} q_{C,\sigma_k} = k$. Since $p_h(\sigma_k)$ belongs to $\mathfrak{N}$, σ_k is of type V or H in C, so $q_{C,\sigma_k} = \alpha$. Since x dominates $\mathfrak{S}_C$, there exists, by [7, Lemma 4.5], β in $\mathfrak{S}_x^k$ that dominates $q_{C,\sigma_k} = \alpha$.

In the third case, choose a v-chain C in $\mathfrak{S}$ with tail σ_{k-1} such that the $\mathfrak{O}$-depth in C of σ_{k-1} is $k-1$. Then $\operatorname{depth}_{\mathfrak{S}_C} q_{C,\sigma_{k-1}} = k-1$. Since $p_h(\sigma_{k-1})$ belongs to $\mathfrak{N}$, σ_{k-1} is of type V or H in C, so $q_{C,\sigma_{k-1}} = p_v(\sigma_{k-1})$. From Lemma 6.3.1 (4), it follows, since $k-1$ is odd, that σ_{k-1} is of type H. Since $p_v(\sigma_{k-1}) > p_h(\sigma_{k-1}) = \alpha$, it follows that $\operatorname{depth}_{\mathfrak{S}_C} p_h(\sigma_{k-1}) \ge k$ (in fact equality

holds as is easily seen). Since x dominates $\mathfrak{S}_C$, there exists, by [7, Lemma 4.5], β in $\mathfrak{S}_x^k$ that dominates $p_h(\sigma_{k-1}) = \alpha$. $\square$

10.2 Proof that $\mathfrak{O}\phi\mathfrak{O}\pi = identity$

Let $\mathfrak{S}$ be a monomial in $\mathfrak{O}\mathfrak{N}$ and let $\mathfrak{O}\pi = (w, \mathfrak{S}')$. We need to show that $\mathfrak{O}\phi$ applied to the pair $(w, \mathfrak{S}')$ gets us back to $\mathfrak{S}$. We know from (3) of Proposition 4.1.1 that w $\mathfrak{O}$-dominates $\mathfrak{S}'$, so $\mathfrak{O}\phi$ can indeed be applied to the pair $(w, \mathfrak{S}')$.

The main ingredients of the proof are the corresponding assertion in the case of Grassmannian [7, Proposition 4.2] and the following claim which we will presently prove:

$$(\mathfrak{S}')_{w,j,j+1} = \mathfrak{S}'_j(\text{up}) \cup \mathfrak{S}'_{j+1}(\text{up}) \quad \text{for every odd integer } j$$

Let us first see how the assertion follows assuming the truth of the claim, by tracing the steps involved in applying $\mathfrak{O}\phi$ to $(w, \mathfrak{S}')$. From the claim it follows that when we partition $\mathfrak{S}'$ into pieces (see §8), we get $\mathfrak{S}'_j(\text{up}) \cup \mathfrak{S}'_{j+1}(\text{up})$ (for odd integers j). Adding the mirror images will get us to $\mathfrak{S}'_j \cup \mathfrak{S}'_{j+1}$. From Corollary 9.3.9 it follows that $w_{j,j+1}$ is exactly the element of $I(d, 2d)$ obtained by acting π on $\mathfrak{S}_j \cup \mathfrak{S}_{j+1}$. Now, since $\phi \circ \pi$ = identity, it follows that on application of ϕ to $(w_{j,j+1}, \mathfrak{S}'_j \cup \mathfrak{S}'_{j+1})$ we obtain $\mathfrak{S}_j \cup \mathfrak{S}_{j+1}$. By twisting the two diagonal elements in $\mathfrak{S}_j \cup \mathfrak{S}_{j+1}$ (if they exist at all) and removing the elements below the diagonal $\mathfrak{d}$, we get back $\mathfrak{S}^{\text{pr}}_{j,j+1}$. Taking the union of $\mathfrak{S}^{\text{pr}}_{j,j+1}$ (over odd integers j), we get back $\mathfrak{S}$.

Thus we need only prove the claim. Since $\mathfrak{S}'$ is the union over all odd integers of the right hand sides (this follows from the definition of $\mathfrak{S}'$), and the left hand sides as j varies are mutually disjoint, it is enough to show that the right hand side is contained in the left hand side. Thus we need only prove: for j an odd integer and α an element in $\mathfrak{S}'_j(\text{up}) \cup \mathfrak{S}'_{j+1}(\text{up})$,

- every v-chain in $\mathfrak{S}'$ with head α is $\mathfrak{O}$-dominated by w^j.
- there exists a v-chain in $\mathfrak{S}'$ with head α that is not $\mathfrak{O}$-dominated by w^{j+2}.

To prove the first item, write $\mathfrak{T} = \mathfrak{S}^{j,j+1} := \{\alpha \in \mathfrak{S} | \mathfrak{O}\text{-depth}_{\mathfrak{S}}(\alpha) \geq j\}$ and set $\mathfrak{O}\pi(\mathfrak{T}) = (x, \mathfrak{T}')$. By Proposition 6.3.6, we have $\mathfrak{T}^{\text{pr}}_i \cup \mathfrak{T}^{\text{pr}}_{i+1} = \mathfrak{S}^{\text{pr}}_{i+j-1} \cup \mathfrak{S}^{\text{pr}}_{i+j}$ for any odd integer i. Thus, by the description of $\mathfrak{O}\pi$, we have $\mathfrak{T}' = \cup_{k \geq j} \mathfrak{S}'_k(\text{up})$. By Corollary 9.3.9 and the description of $\mathfrak{O}\pi$, we have $x = w^j$. By Corollary 9.3.10, any v-chain in $\mathfrak{S}'$ with head belonging to $\mathfrak{S}'_j(\text{up}) \cup \mathfrak{S}'_{j+1}(\text{up})$ is contained entirely in $\cup_{k \geq j} \mathfrak{S}'_k(\text{up})$. Finally, by Proposition 4.1.1 (3) applied to $\mathfrak{T}$, the desired conclusion follows.

To prove the second item we use Lemma 9.4.3. Proceed by decreasing induction on j. For j sufficiently large the assertion is vacuous, for $\mathfrak{S}'_j(\text{up}) \cup \mathfrak{S}'_{j+1}(\text{up})$ is empty. To prove the induction step, assume that the assertion holds for $j + 2$. If the v-chain consisting of the single element α is not $\mathfrak{O}$-dominated by w^{j+2}, then we are done. So let us assume the contrary. Since the $\mathfrak{O}$-depth of the singleton v-chain α is at most 2, it follows from Lemma 9.2.2 that $w_{j+2,j+3}$ $\mathfrak{O}$-dominates the v-chain α. Apply Lemma 9.4.3 with $k = j + 2$. By its conclusion,

either there exists $\beta \in \mathfrak{S}'_{j+2}(\text{up})$ such that $\alpha > \beta$ or there exists $\gamma \in \mathfrak{S}'_{j+3}(\text{up})$ such that $p_h(\alpha) > \gamma$.

First suppose that a γ as above exists. By induction, there exists a v-chain in $\mathfrak{S}'$—call it D—with head γ that is not $\mathfrak{O}$-dominated by w^{j+4}. Let C be the concatenation of $\alpha > \gamma$ and D. Since elements of D have $\mathfrak{O}$-depth at least 3 in C (Lemma 6.3.1 (1)), it follows from Corollary 9.2.3 that C is not $\mathfrak{O}$-dominated by w^{j+2}, and we are done.

Now suppose that such a γ does not exist. Then a β as above exists. If $\alpha > \beta$ is not $\mathfrak{O}$-dominated by w^{j+2} we are again done. So assume the contrary. Since the $\mathfrak{O}$-depth of β in $\alpha > \beta$ is at least 2, it follows that there exists an element of $(\mathfrak{S}_w)_{j+3}$ that dominates β. Applying Lemma 9.4.3 again, this time with $k = j+3$, we find $\gamma' \in \mathfrak{S}'_{j+3}(\text{up})$ such that $\beta > \gamma'$. Arguing as in the previous paragraph with γ' in place of γ, we are done. □

10.3 Proof that $\mathfrak{O}\pi\mathfrak{O}\phi = \mathit{identity}$

Let $\mathfrak{T}$ be a monomial in $\mathfrak{O}\mathfrak{N}$ and w an element of $I(d)$ that $\mathfrak{O}$-dominates $\mathfrak{T}$. We can apply $\mathfrak{O}\phi$ to the pair $(w, \mathfrak{T})$ to obtain a monomial $\mathfrak{T}^\star_w$ in $\mathfrak{O}\mathfrak{N}$. We need to show that $\mathfrak{O}\pi$ applied to $\mathfrak{T}^\star_w$ results in $(w, \mathfrak{T})$. The main step of the proof is to establish the following:

$$\mathfrak{T}^\star_{w,j,j+1} = (\mathfrak{T}^\star_w)^{\text{pr}}_{j,j+1} \tag{10.3.1}$$

(for the meaning of the left and right sides of the above equation, see §8 and §7 respectively). Assuming this for the moment let us show that $\mathfrak{O}\pi \circ \mathfrak{O}\phi =$ identity.

We trace the steps involved in applying $\mathfrak{O}\pi$ to $\mathfrak{T}^\star_w$. From Eq. (10.3.1) it follows that when we break up $\mathfrak{T}^\star_w$ according to the $\mathfrak{O}$-depths of its elements as in §7, we get $\mathfrak{T}^\star_{w,j,j+1}$ (as j varies over odd integers). The next step in the application of $\mathfrak{O}\pi$ is the passage from $(\mathfrak{T}^\star_w)^{\text{pr}}_{j,j+1}$ to $(\mathfrak{T}^\star_w)_{j,j+1}$. This involves replacing σ_j by its projections and adding the mirror image of the remaining elements of $(\mathfrak{T}^\star_w)^{\text{pr}}_{j,j+1}$. It follows from Proposition 8.2.2 (3) that $\sigma_j = \delta_j$ and so $(\mathfrak{T}^\star_w)_{j,j+1} = (\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^\#_{w,j,j+1})^\star$. The next step is to apply π to $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^\#_{w,j,j+1})^\star$. Since π is the inverse of ϕ (as proved in [7]), we have $\pi((\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^\#_{w,j,j+1})^\star) = (w_{j,j+1}, \mathfrak{T}_{w,j,j+1})$. Since $\mathfrak{S}_w$ and $\mathfrak{T}$ are respectively the unions, as j varies over odd integers, of $(\mathfrak{S}_w)_{j,j+1}$ and $\mathfrak{T}_{w,j,j+1}$, we see that $\mathfrak{O}\phi$ applied to $\mathfrak{T}^\star_w$ results in $(w, \mathfrak{T})$.

Thus it remains only to establish Eq. (10.3.1). It is enough to show that the left hand side is contained in the right hand side, for the union over all odd j of either side is $\mathfrak{T}^\star_w$ and the right hand side is moreover a disjoint union. In other words, we need only show that the $\mathfrak{O}$-depth in $\mathfrak{T}^\star_w$ of an element of $\mathfrak{T}^\star_{w,j,j+1}$ is either j or $j+1$. We will show, more precisely, that, for any element β of $\mathfrak{S}_w$, the $\mathfrak{O}$-depth in $\mathfrak{T}^\star_w$ of any element of $\mathfrak{O}\mathfrak{P}^*_\beta$ equals the depth in $\mathfrak{S}_w$ of β. Lemma 9.4.4 will be used for this purpose.

Let α be an element of $\mathfrak{O}\mathfrak{P}^*_\beta$ and set $e := \mathfrak{O}\text{-depth}_{\mathfrak{T}^\star_w}(\alpha)$. We first show, by induction on $d := \text{depth}_{\mathfrak{S}_w}\beta$, that $e \geq d$. There is nothing to prove in case $d = 1$, so we proceed to the induction step. Let β' be the element of $\mathfrak{S}_w$ of depth

$d-1$ such that $\beta' > \beta$. If there exists α' in $\mathfrak{O}\mathfrak{P}^\star_{\beta'}$ with $\alpha' > \alpha$, the desired conclusion follows from Corollary 6.1.3 (3) and induction. Lemma 9.4.4 says that such an α' exists in case d is even. So suppose that d is odd and such an α' does not exist. The same lemma now says that $p_h(\delta_{d-2}) > \alpha$, so the desired conclusion follows from Lemma 6.3.1 (1).

We now show, by induction on e, that $d \geq e$. There is nothing to prove in case $e = 1$, so we proceed to the induction step. Let C be a v-chain in $\mathfrak{T}^\star_w$ with tail α and having the good property of Proposition 6.3.3. Let α' be the immediate predecessor in C of α. Let β' in $\mathfrak{S}_w$ be such that $\alpha' \in \mathfrak{O}\mathfrak{P}^*_{\beta'}$ (we are not claiming at the moment that β' is unique although that is true and follows from the assertion that we are proving, the distinguishedness of $\mathfrak{S}_w$, and the fact that β' dominates α'). It follows from Corollary 9.4.6 that $\beta' > \beta$.

Let $d' := \mathrm{depth}_{\mathfrak{S}_w}\beta'$. It follows from Corollary 6.1.3 (3) that $e' < e$ where $e' := \mathfrak{O}\text{-depth}_{\mathfrak{T}^\star_w}(\alpha')$. We have, $d \geq d'+1 \geq e'+1 \geq (e-2)+1 = e-1$, the first equality being justified because $\beta' > \beta$, the second by the induction hypothesis, and the last by Lemma 6.3.1 (1). It suffices to rule out the possibility that $d = e-1$. So assume $d = e-1$. Then $d = d'+1$ and $d' = e' = e-2$. It follows from (1) of Lemma 6.3.1 that the v-chain $\alpha' > \alpha$ has $\mathfrak{O}$-depth 3 and from (3) of the same lemma that e' is odd. But then we get a contradiction to Proposition 8.2.2 (2) (α' and α belong to $\mathfrak{T}^\star_{w,d',d'+1}$). The proof of Eq. (10.3.1) is thus over. $\square$

10.4 Proof of Proposition 4.1.3

Observe that the condition (‡) makes sense also for a monomial of $\mathfrak{N}$. By virtue of belonging to $I(d)$, v has f^* as an entry. It follows from the description of the bijection $w \leftrightarrow \mathfrak{S}_w$ of §5.1.2 that for an element w of $I(d)$ to satisfy (‡) it is necessary and sufficient that $\mathfrak{S}_w$ (equivalently all its parts $\mathfrak{S}_{w,j,j+1}$) satisfy (‡).

(1) Since $\mathfrak{T}$ satisfies (‡), so do its parts $\mathfrak{T}_{w,j,j+1}$ and $\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^\#_{w,j,j+1}$ (adding the mirror image preserves (‡)). Since $\mathfrak{S}_{w,j,j+1}$ also satisfies (‡), it follows from the description of the map ϕ of [7] (observe the passage from a piece $\mathfrak{P}$ to its "star" $\mathfrak{P}^*$) that the $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^\#_{w,j,j+1})^\star$ satisfy (‡). Since the "twisting" involved in the passage from $(\mathfrak{T}_{w,j,j+1} \cup \mathfrak{T}^\#_{w,j,j+1})^\star$ to $\mathfrak{T}^\star_{w,j,j+1}$ involves only a rearrangement of row and column indices, it follows that the $\mathfrak{T}^\star_{w,j,j+1}$ satisfy (‡). Finally so also does their union $\mathfrak{T}^\star_w$.

(2) The parts $\mathfrak{S}^{\mathrm{pr}}_{j,j+1}$ of $\mathfrak{S}$ clearly satisfy (‡). Therefore so do the $\mathfrak{S}_{j,j+1}$, for, first of all, adding the mirror image preserves (‡), and then the removal of σ_j and addition of its projections involves only a rearrangement of row and column indices. It follows from description of the map π of [7] (observe the passage from a block $\mathfrak{B}$ to the pair $(w(\mathfrak{B}), \mathfrak{B}'))$ that both $\mathfrak{S}_{w,j,j+1}$ and $\mathfrak{S}'_{j,j+1}$ satisfy (‡). Finally, $\mathfrak{S}_w$ and $\mathfrak{S}'$ being the union (respectively) of $\mathfrak{S}_{w,j,j+1}$ and $\mathfrak{S}'_{j,j+1}(\mathrm{up})$, they satisfy (‡). $\square$

Part IV
An Application

As an application of the main theorem (Theorem 2.3.1), an interpretation of the multiplicity is presented.

11 Multiplicity counts certain paths

Fix elements v, w in $I(d)$ with $v \leq w$. It follows from Corollary 2.3.2 that the multiplicity of the Schubert variety $X(w)$ in $\mathfrak{M}_d(V)$ at the point $\mathfrak{e}^v$ can be interpreted as the cardinality of a certain set of non-intersecting lattice paths. We first illustrate this by means of two examples and then justify the interpretation.

11.1 Description and illustration

The points of $\mathfrak{N}$ can be represented, in a natural way, as the lattice points of a grid. The column indices of the points of the grid are the entries of v and the row indices are the entries of $\{1, \ldots, 2d\} \setminus v$. In Figure 11.1.1 the points of $\mathfrak{O}\mathfrak{N}$ and those of the diagonal in $\mathfrak{N}$ are shown (for the specific choice of v in Example 11.1.1). The open circles represent the points of $\mathfrak{S}_w(\text{up})$, where $\mathfrak{S}_w$ is the distinguished monomial in $\mathfrak{N}$ that is associated to w as in §5.1.2. From each point β of $\mathfrak{S}_w(\text{up})$ we draw a vertical line upwards from β and let $\beta(\text{start})$ denote the top most point of $\mathfrak{O}\mathfrak{N}$ on this line. In case β is not on the diagonal, draw also a horizontal line rightwards from β and let $\beta(\text{finish})$ denote the right most point of $\mathfrak{O}\mathfrak{N}$ on this line. In case β is on the diagonal, then $\beta(\text{finish})$ is not a fixed point but varies subject to the following constraints:

- $\beta(\text{finish})$ is one step away from the diagonal (that is, it is of the form (r, c), for some entry c of v, where r is the largest integer less than c^* that is not an entry of v);
- the column index of $\beta(\text{finish})$ is not less than that of β;
- if $\text{depth}_{\mathfrak{S}_w}\beta$ is odd, then the horizontal projection of $\beta(\text{finish})$ is the same as the vertical projection of $\gamma(\text{finish})$ where γ is the diagonal element of $\mathfrak{S}_w$ of depth 1 more than that of β.

With v and w as in Example 11.1.1, we have $\beta(\text{start}) = (6, 3)$ and $\beta(\text{finish}) = (9, 5)$ for $\beta = (9, 3)$; $\beta(\text{start}) = \beta(\text{finish}) = (21, 20)$ for $\beta = (21, 20)$; $\beta(\text{start}) = (15, 11)$ for the diagonal element $\beta = (36, 11)$; $\beta(\text{start}) = (6, 1)$ for the diagonal element $\beta = (46, 1)$. In the particular case (of non-intersecting lattice paths) drawn in Figure 11.1.1, $\beta(\text{finish}) = (27, 19)$ for $\beta = (36, 11)$ and $\beta(\text{finish}) = (28, 14)$ for $\beta = (46, 1)$.

A *lattice path* between a pair of such points $\beta(\text{start})$ and $\beta(\text{finish})$ is a sequence $\alpha_1, \ldots, \alpha_q$ of elements of $\mathfrak{O}\mathfrak{N}$ with $\alpha_1 = \beta(\text{start})$ and $\alpha_q = \beta(\text{finish})$

such that, for $1 \leq j \leq q-1$, if we let $\alpha_j = (r, c)$, then α_{j+1} is either (R, c) or (r, C) where R is the least element of $\{1, \ldots, 2d\} \setminus v$ that is bigger than r and C the least element of v that is bigger than c. Note that if $\beta(\text{start}) = (r, c)$ and $\beta(\text{finish}) = (R, C)$, then q equals

$$|(\{1, \ldots, 2d\} \setminus v) \cap \{r, r+1, \ldots, R\}| + |v \cap \{c, c+1, \ldots, C\}| - 1,$$

where $|\cdot|$ is used to denote cardinality.

Consider the set $\mathfrak{Paths}^w$ of all tuples $(\Lambda_\beta)_{\beta \in \mathfrak{S}_w(\text{up})}$ of paths where

- Λ_β is a lattice path between $\beta(\text{start})$ and $\beta(\text{finish})$ (if β is on the diagonal, then $\beta(\text{finish})$ is allowed to vary in the manner described above);
- Λ_β and Λ_γ do not intersect for $\beta \neq \gamma$.

The number of such p-tuples, where $p := |\mathfrak{S}_w(\text{up})|$, is the multiplicity of $X(w)$ at the point $\mathfrak{e}^v$.

Example 11.1.1 Let $d = 23$,

$$v = (1,2,3,4,5,11,12,13,14,19,20,22,23,26,29,30,31,32,37,38,39,40,41),$$
$$w = (4,5,9,10,14,17,18,21,23,25,27,28,31,32,34,35,36,39,40,41,44,45,46),$$

so that

$$\begin{aligned}\mathfrak{S}_w =& \{(9,3),(10,2),(17,13),(18,12),(21,20),(25,22),(27,26),\\ &(28,19),(34,30),(35,29),(36,11),(44,38),(45,37),(46,1)\}\end{aligned}$$

and $\mathfrak{S}_w(\text{up}) =$

$$\{(9,3),(10,2),(17,13),(18,12),(21,20),(25,22),(28,19),(36,11),(46,1)\}.$$

A particular element of $\mathfrak{Paths}^w$ is depicted in Figure 11.1.1. $\square$

Example 11.1.2 Figure 11.1.2 shows all the elements of $\mathfrak{Paths}^w$ in the following simple case:

$$d = 7, \quad v = (1,2,3,4,7,9,10), \quad \text{and} \quad w = (4,6,7,10,12,13,14).$$

We have $\mathfrak{S}_w = \{(6,3),(12,9),(13,2),(14,1)\}$, $\mathfrak{S}_w(\text{up}) = \{(6,3),(13,2)(14,1)\}$. There are 15 elements in $\mathfrak{Paths}^w$ and thus the multiplicity in this case is 15. $\square$

Example 11.1.3 Let $d = 10$,

$$v = (1,2,3,4,6,8,11,12,14,16), \quad \text{and} \quad w = (8,9,11,14,15,16,17,18,19,20).$$

so that $\mathfrak{S}_w = \{(20,1)(19,2)(18,3),(17,4),(9,6)(15,12)\}$. Figure 11.1.3 shows a tuple of paths that is disallowed (meaning one that is not in $\mathfrak{Paths}^w$). The elements of $\mathfrak{O}\mathfrak{N}$ are represented as usual by a grid. The slanted line represents

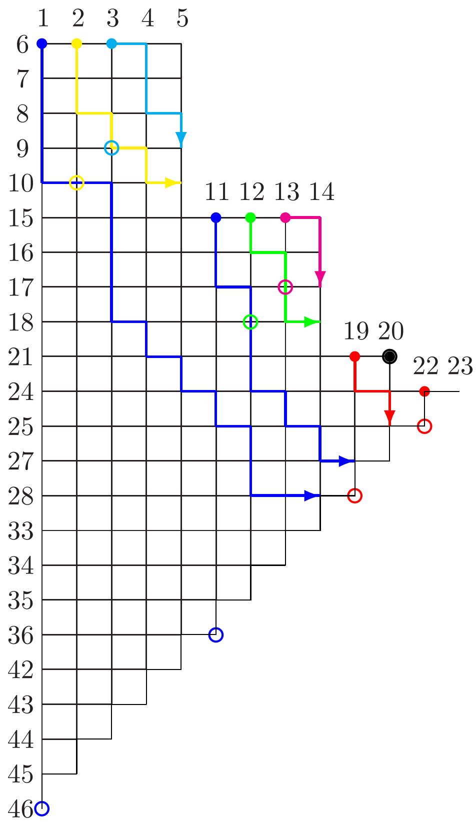


Figure 11.1.1: An element of $\mathfrak{Paths}^w$ with v and w as in Example 11.1.1

the diagonal $\mathfrak{d}$. The solid dot represents the point of $\mathfrak{S}_w(\mathrm{up})$ that is not on $\mathfrak{d}$, and the crosses on $\mathfrak{d}$ represent the points of $\mathfrak{S}_w(\mathrm{up})$ that lie on $\mathfrak{d}$. The tuple is disallowed because the horizontal projection of the last point of the path Λ_{β_1} is not the vertical projection of the last point of the path Λ_{β_2}, where $\beta_1 = (20, 1)$ and $\beta_2 = (19, 2)$ are the diagonal elements of $\mathfrak{S}_w$ of depths 1 and 2 respectively. □

11.2 Justification for the interpretation

We now justify the interpretation in the previous subsection of the multiplicity. Corollary 2.3.2 says that the multiplicity is the number of monomials in $\mathfrak{O}\mathfrak{R}$ of maximal cardinality that are square-free and $\mathfrak{O}$-dominated by w. Any such monomial contains $\mathfrak{O}\mathfrak{R} \setminus \mathfrak{O}\mathfrak{N}$, for, by the definition of $\mathfrak{O}$-domination, adding or removing elements of $\mathfrak{O}\mathfrak{R} \setminus \mathfrak{O}\mathfrak{N}$ to or from a monomial does not alter the status of its $\mathfrak{O}$-domination. One could therefore equally well consider the number of monomials in $\mathfrak{O}\mathfrak{N}$ of maximal cardinality that are square-free and $\mathfrak{O}$-dominated

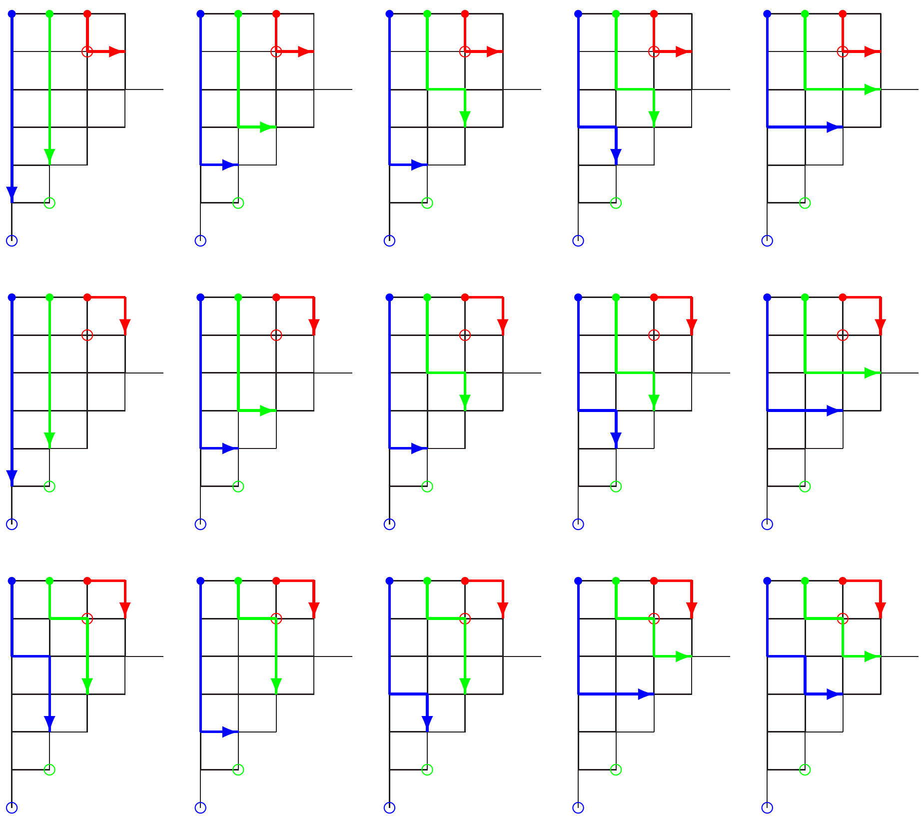


Figure 11.1.2: All the 15 non-intersecting lattice paths of Example 11.1.2

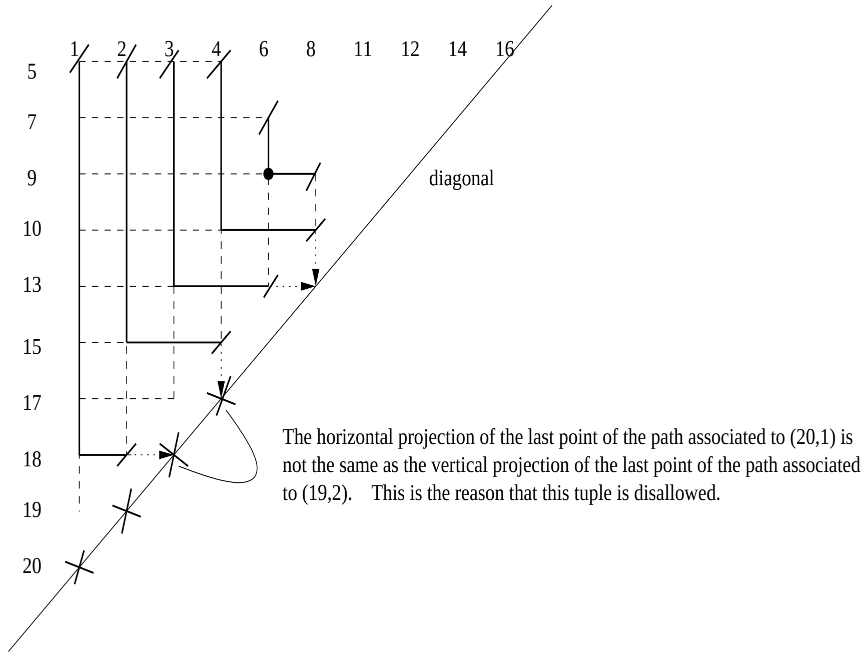


Figure 11.1.3: A disallowed tuple of lattice paths (see Example 11.1.3)

by w. We now establish a bijection between the set $\mathfrak{Mon}^w$ of such monomials and the set $\mathfrak{Paths}^w$ of non-intersecting lattice paths as in §11.1.

Each element Λ of $\mathfrak{Paths}^w$ can be thought of, in the obvious way, as a monomial in $\mathfrak{O}\mathfrak{N}$. We will continue to denote the corresponding monomial by Λ. It is clear that the monomial Λ is square-free and that all such monomials Λ have the same cardinality (in particular, that if $\Lambda_1 \subseteq \Lambda_2$ for two such monomials then $\Lambda_1 = \Lambda_2$). In order to establish the bijection it therefore suffices to prove the following proposition.

Proposition 11.2.1 *1. w is the element of $I(d)$ obtained on application of $\mathfrak{O}\pi$ to the monomial Λ (in particular (see Proposition 4.1.1), the monomial Λ is $\mathfrak{O}$-dominated by w).*

2. Given a monomial $\mathfrak{T}$ of $\mathfrak{O}\mathfrak{N}$ that is square-free and $\mathfrak{O}$-dominated by w, there exists Λ such that $\mathfrak{T} \subseteq \Lambda$.

Proof: (1) Write $\Lambda = (\Lambda_\beta)_{\beta \in \mathfrak{S}_w(\mathrm{up})}$. From the description of the map $\mathfrak{O}\pi$ in §7, it follows that it suffices to show that Λ_k^{pr} (in the notation of §7) is the union $\cup\Lambda_\beta$ where β runs over all elements of depth k in $\mathfrak{S}_w(\mathrm{up})$. In other words, it suffices to show that the $\mathfrak{O}$-depth in Λ of any element of Λ_β equals the depth in $\mathfrak{S}_w$ of β. To prove this, we observe the following (these assertions are easily seen to be true thinking in terms of pictures): for fixed $\beta \in \mathfrak{S}_w(\mathrm{up})$ and $\alpha \in \Lambda_\beta$,

(A) For β' in $\mathfrak{S}_w(\text{up})$ such that $\beta' > \beta$, there exists $\alpha' \in \Lambda_{\beta'}$ such that $\alpha' > \alpha$.

(B) If $\alpha' > \alpha$ for some α' in $\Lambda_{\beta'}$ for some β' in $\mathfrak{S}_w(\text{up})$, then $\beta' > \beta$. If, furthermore, β and β' are diagonal, their depths in $\mathfrak{S}_w$ are 1 apart, and the depth in $\mathfrak{S}_w$ of β is even, then the following is not possible: $p_h(\alpha')$ belongs to $\mathfrak{N}$ and $p_h(\alpha') > \alpha$.

From (A) it is immediate that the $\mathfrak{O}$-depth e in Λ of an element α of Λ_β is not less than the depth d in $\mathfrak{S}_w$ of β. We now show, by induction on e, that $e \leq d$. For $e = 1$ there is nothing to show. Suppose that $e \geq 2$. Let C be a v-chain in Λ having tail α and the good property of Proposition 6.3.3, α' the immediate predecessor in C of α, e' the $\mathfrak{O}$-depth of α' in Λ, β' the element of $\mathfrak{S}_w(\text{up})$ such that $\alpha' \in \Lambda_{\beta'}$, and d' the depth in $\mathfrak{S}_w$ of β'. From Corollary 6.1.3 (3) it follows that $e' \leq e - 1$, so we may apply induction. From (B) it follows that $d' \leq d - 1$, so that, by induction, $e' \leq d - 1$. If $e' \leq d - 2$, then we are done by Lemma 6.3.1 (1). So suppose that $e' = d' = d - 1$. If d is odd, then the conclusion $e \leq d$ follows from (1) and (3) of the same lemma. In case d is even, then it follows from condition (B) and (1) of the same lemma.

(2) Let $\mathfrak{T}$ be a square-free monomial in $\mathfrak{O}\mathfrak{N}$ that is $\mathfrak{O}$-dominated by w. To construct Λ such that $\mathfrak{T} \subseteq \Lambda$, we construct the "components" Λ_β. As in §8, let $\mathfrak{P}_\beta$ denote the piece of $\mathfrak{T}$ corresponding to $\beta \in \mathfrak{S}_w$. From every point belonging to $\mathfrak{P}_\beta(\text{up})$ and also from $\beta(\text{start})$ carve out the South-West quadrant; if β is not diagonal, then do this also from $\beta(\text{finish})$. The boundary of the carved out portion (intersected with $\mathfrak{O}\mathfrak{N}$) gives a lattice path starting from $\beta(\text{start})$. In case β is not diagonal, the path ends in $\beta(\text{finish})$. In this case as well as in the case when β is diagonal and of even depth in $\mathfrak{S}_w$, we take Λ_β to be this lattice path. In case β is diagonal and of odd depth in $\mathfrak{S}_w$ we do the carving out from one more point before taking Λ_β to be the boundary of the carved out region, namely from the point that is one step away from the diagonal and whose horizontal projection is the vertical projection of the end point of Λ_γ where γ is the diagonal element of $\mathfrak{S}_w$ of depth 1 more than β. We need to justify why carving out from the extra point is still valid, and we do this now by applying Lemma 8.1.4.

Let us first choose notation that is consistent with that of that lemma. Let β and γ be diagonal elements in $\mathfrak{S}_w$ of depths d and $d+1$. Assume that d is odd. Let the pieces of $\mathfrak{T}$ corresponding to β and γ, when their elements are arranged in increasing order of row and column indices, look like this:

$$\ldots,\ (r_1, a^*),\ (a, r_1^*),\ \ldots; \qquad \ldots,\ (r_2, b^*),\ (b, r_2^*),\ \ldots$$

It is easy to see that the conditions on the numbers in the above display that provide the requisite justification are: $r_1 \leq b$ and $a^* < b^*$ (if $\mathfrak{P}_\beta$ is empty then the justification is easy). To prove that $a^* < b^*$, observe that the diagonal elements in $\mathfrak{P}_\beta^*$ and $\mathfrak{P}_\gamma^*$ are respectively (a, a^*) and (b, b^*), and apply Lemma 8.1.3 (2). That $r_1 \leq b$ now follows from Lemma 8.1.4 (1). This finishes the justification.

It suffices to prove the following claim: the lattice paths Λ_β as β varies are non-intersecting. Suppose that Λ_β and $\Lambda_{\beta'}$ intersect for $\beta \neq \beta'$. Let α be a point

of intersection. Clearly β dominates all elements of Λ_β and in particular α; for the same reason β' also dominates α. By the distinguishedness of $\mathfrak{S}_w$, we may assume without loss of generality that $\beta' > \beta$. It is easy to see graphically that if γ in $\mathfrak{S}_w$ is such that $\beta' > \gamma > \beta$ then Λ_γ intersects either $\Lambda_{\beta'}$ or Λ_β: consider the open portion of $\mathfrak{O}\mathfrak{N}$ "caught between" the segment of $\Lambda_{\beta'}$ from β'(start) to α and the segment of Λ_β from β(start) to α; the starting point γ(start) of Λ_γ lives in this region but its ending point does not (points strictly to the Northwest of α can neither be of the form γ(finish) for γ not on the diagonal nor can they be one step away from the diagonal); so Λ_γ must intersect one of the two lattice path segments. We may therefore assume that the depths of β' and β differ by 1.

We now apply Lemma 9.4.4. From the construction of Λ_β it readily follows that α satisfies the hypotheses (a), (b), and (c) of that lemma. By the conclusion of Lemma 9.4.4, there exists $\alpha' \in \mathfrak{P}^*_{\beta'}(\mathrm{up})$ such that $\alpha' > \alpha$. On the other hand, it follows from the construction of $\mathfrak{P}^*_{\beta'}$ from $\mathfrak{P}_{\beta'}$, and from the construction of $\Lambda_{\beta'}$ that two elements one from $\mathfrak{P}^*_{\beta'}$ and another from $\Lambda_{\beta'}$ are not comparable. This is a contradiction to the comparability of α' and α. $\square$

References

[1] M. Brion and P. Polo, *Generic singularities of certain Schubert varieties*, Math. Z., **231**, no. 2, 1999, pp. 301–324.

[2] E. De Negri, *Some results on Hilbert series and a-invariant of Pfaffian ideals*, Math. J. Toyama Univ., **24**, 2001, pp. 93–106.

[3] S. R. Ghorpade and C. Krattenthaler, *The Hilbert series of Pfaffian rings*, in: *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000)*, Springer, Berlin, 2004, pp. 337–356.

[4] S. R. Ghorpade and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in the Symplectic Grassmannian*, Trans. Amer. Math. Soc., **358**, 2006, pp. 5401–5423.

[5] J. Herzog and N. V. Trung, *Gröbner bases and multiplicity of determinantal and Pfaffian ideals*, Adv. Math., **96**, no. 1, 1992, pp. 1–37.

[6] T. Ikeda and H. Naruse, *Excited Young diagrams and equivariant Schubert calculus*, arXiv:math/0703637.

[7] V. Kodiyalam and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in Grassmannians*, J. Algebra, **270**, no. 1, 2003, pp. 28–54.

[8] C. Krattenthaler, *On multiplicities of points on Schubert varieties in Grassmannians. II*, J. Algebraic Combin., **22**, no. 3, 2005, pp. 273–288.

[9] V. Kreiman, *Monomial bases and applications for Schubert and Richardson varieties in ordinary and affine Grassmannians*, Ph. D. Thesis, Northeastern University, 2003.

[10] V. Kreiman, *Local Properties of Richardson Varieties in the Grassmannian via a Bounded Robinson-Schensted-Knuth Correspondence*, preprint, 2005, URL `arXiv:math.AG/0511695`.

[11] V. Kreiman and V. Lakshmibai, *Multiplicities of singular points in Schubert varieties of Grassmannians*, in: *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000)*, Springer, Berlin, 2004, pp. 553–563.

[12] V. Kreiman and V. Lakshmibai, *Richardson varieties in the Grassmannian*, in: *Contributions to automorphic forms, geometry, and number theory*, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 573–597.

[13] V. Lakshmibai and C. S. Seshadri, *Geometry of G/P. V*, J. Algebra, **100**, no. 2, 1986, pp. 462–557.

[14] V. Lakshmibai and J. Weyman, *Multiplicities of points on a Schubert variety in a minuscule G/P*, Adv. Math., **84**, no. 2, 1990, pp. 179–208.

[15] P. Littelmann, *Contracting modules and standard monomial theory for symmetrizable Kac-Moody algebras*, J. Amer. Math. Soc., **11**, no. 3, 1998, pp. 551–567.

[16] K. N. Raghavan and S. Upadhyay, *Initial ideals of tangent cones to Schubert varieties in orthogonal Grassmannians*. In preparation.

[17] C. S. Seshadri, *Geometry of G/P. I. Theory of standard monomials for minuscule representations*, in: *C. P. Ramanujam—a tribute*, vol. 8 of Tata Inst. Fund. Res. Studies in Math., Springer, Berlin, 1978, pp. 207–239.

Index